SUPPLEMENT

TO

PROFESSOR LORGNA'S

SUMMATION OF SERIES.

TO WHICH ARE ADDED,

REMARKS

ON

MR. LANDEN'S OBSERVATIONS ON THE SAME SUBJECT.

BY THE TRANSLATOR OF THE ABOVE WORK,

HENRY CLARKE.

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ADVERTISEMENT.

UPON reviſing my tranſlation of Mr. Lorgna's Treatiſe on Series immediately after its publication, I found that ſome additions and alterations were neceſſary to render it complete. Alſo, that in the Appendix, I had omitted ſeveral curious and uſeful formulæ, ſome of which were originally intended to have been inſerted therein, and others that have ſince been inveſtigated. Not that any other principles have been diſcovered, beſides thoſe pointed out in the Appendix, whereby this branch of the Mathematics may be farther extended; but only the application of certain and peculiar proceſſes therein. Indeed no new principles ſeem to be wanting (ſuppoſing any could be found) for, not to mention the well-known methods peculiarly adapted to this purpoſe, every one knows that a numberleſs variety of ſeries with their ſummations will be obtained by barely expanding fractional or radical, binomial, trinomial, &c. expreſſions; which being again differently combined, produce a variety of other ſeries. It is alſo as well known that by the application of the fluxionary *calculus* to this ſubject we may produce an infinite variety of ſeries that are ſummible; the only addreſs or artifice requiſite herein, being to aſſume ſuch expreſſions that when they have undergone the various operations of multiplying, &c. by the fluxion, or other functions, of the indefinite quantity, neceſſary to reduce their expanded values to the form of the ſeries propoſed, the fluents of the reſulting expreſſions may be attainable without infinite ſeries; that is, either algebraically, or by means of the conic ſections.

I had intended, in this Supplement, to have given the inveſtigations or demonſtrations of the formulæ for ſummation in the different orders of ſeries in the Appendix, under the names of their reſpective authors; but finding it would far exceed the limits I had propoſed, and the utility thereof not appearing very great, as the names of thoſe authors were before ſpecified, I have, at preſent, relinquiſhed the deſign.

To

To offer any apology for the typographical errors which have eſcaped in this Treatiſe, would, I apprehend, be needleſs; as every reader muſt be acquainted with the difficulties attending the corrections of the ſheets in a ſubject ſo intricate to a compoſitor, when the author's ſituation or diſtance from the preſs renders it impracticable for him to inſpect the reviſes. Some apology however may, perhaps, be thought neceſſary for the freedom I have uſed in this Supplement reſpecting Mr. Landen's *Obſervations*, &c. In defence of which I can only ſay, that it did not ariſe from any private pique or perſonal reſentment towards that Gentleman, for the treatment he has thought proper to ſhew me on this occaſion; but from a full and clear conviction of the truth of what I have endeavoured to vindicate. And as truth has been my object in this endeavour, I would wiſh every reader who may not fully comprehend the ſubject, not to be implicitly led on the one hand by my bare aſſertions, nor on the other by the ſuggeſtions of thoſe whom envy or prejudice may have prepoſſeſſed; but to ſuſpend his judgement till he is thoroughly acquainted with the principles, and can determine for himſelf. For, as all men are liable to error, I would not, by a too preſumptuous confidence aſſert, that *all* I have advanced on this ſubject is indiſputable; I ſhall therefore be always ready, on conviction, to acknowledge my miſtakes, and ſhall welcome truth whatever quarter it may come from. After all, let not the words of Horace be forgot,

Sunt delicta tamen, quibus ignoviſſe velimus.

A SUP-

A SUPPLEMENT, &c.

262. (F) $$\frac{x^{\frac{p}{q}+1}}{p+q} \pm \frac{x^{\frac{p}{q}+2}}{p+2q} + \frac{x^{\frac{p}{q}+3}}{p+3q} \pm \frac{x^{\frac{p}{q}+4}}{p+4q} + \&c.$$

(Σ) $$\int \frac{x^{\frac{p}{q}}\dot{x}}{q \cdot 1 \mp x}.$$

1. When $\frac{p}{q}$ is a positive whole number, and the signs of the series are $+$, the sum is expressed by $\frac{1}{q} \times : -\mathrm{L}.\overline{1-x} - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c. - \frac{x^{\frac{p}{q}}}{p:q}$.

2. If the signs be $+$ and $-$ alternately, and $\frac{p}{q}$ a positive whole number, the expression for the sum becomes $\frac{1}{q} \times : \pm \mathrm{L}.\overline{1+x} \mp x \pm \frac{x^2}{2} \mp \frac{x^3}{3} \pm \&c. + \frac{x^{\frac{p}{q}}}{p:q}$; the upper or under sign obtaining according as $\frac{p}{q}$ is even or odd.

3. When x is $= 1$, the sum in the former case will be infinite; but in the latter it will be $\frac{1}{q} \times : \pm .6931472 \mp 1 \pm \frac{1}{2} \mp \frac{1}{3} \pm \&c. \frac{1}{p:q}$.

4. And here we may observe, that the formulæ to N° 271. inclusive, are each supposed to be generated while x from 0 attains the value denoted by X; from whence the corrections that may be necessary in the formulæ (Σ), when they are not purely algebraical, will be obvious.

263. (F) $\frac{x^{\frac{m}{n}+1}}{p+q\,.\,m+n} \pm \frac{x^{\frac{m}{n}+2}}{p+2q\,.\,m+2n} + \frac{x^{\frac{m}{n}+3}}{p+3q\,.\,m+3n} \pm$ &c.

$$(\Sigma)\quad \frac{1}{nq} \times : \frac{X^{\pi}}{\pi}\int \frac{x^{\frac{p}{q}}\dot{x}}{1 \mp x} - \frac{1}{\pi}\int \frac{x^{\frac{m}{n}}\dot{x}}{1 \mp x}.$$

When $\pi\left(\frac{m}{n} - \frac{p}{q}\right)$ is an affirmative whole number, and (X) the ultimate value of x such that $X^{\pi} = 1$, the expreſſion is reducible to algebraic quantities, the ſigns of the ſeries being $+$. If the ſigns of the ſeries be alternately $+$ and $-$, π muſt be an *even* poſitive number. See theorem p. 18, when x is $= 1$.

264. (F) $\frac{x^{\frac{r}{s}+1}}{p+q\,.\,m+n\,.\,r+s} \pm \frac{x^{\frac{r}{s}+2}}{p+2q\,.\,m+2n\,.\,r+2s} + \frac{x^{\frac{r}{s}+3}}{p+3q\,.\,m+3n\,.\,r+3s} \pm$ &c.

$$(\Sigma)\quad \frac{1}{nqs\,.\,\pi-\omega} \times : \frac{X^{\pi}}{\pi}\int \frac{x^{\frac{p}{q}}\dot{x}}{1 \mp x} - \frac{1}{\pi}\int \frac{x^{\frac{m}{s}}\dot{x}}{1 \mp x} - \frac{X^{\omega}}{\omega}\int \frac{x^{\frac{m}{n}}\dot{x}}{1 \mp x} - \frac{1}{\omega}\int \frac{x^{\frac{r}{s}}\dot{x}}{1 \mp x}.$$

When $\pi\left(\frac{r}{s} - \frac{p}{q}\right)$ and $\omega\left(\frac{r}{s} - \frac{m}{n}\right)$ are affirmative whole numbers, and the value of X ſuch that $\frac{X^{\pi}}{\pi} - \frac{X^{\omega}}{\omega} = \frac{1}{\pi} - \frac{1}{\omega}$, the expreſſion is algebraical; the ſigns of the ſeries being affirmative. If the ſigns of the ſeries be alternately $+$ and $-$, π and ω muſt be *even* affirm. numbers. See theorems p. 42. when x is $= 1$.

265. (F) $\frac{x^{\frac{t}{u}+1}}{p+q\,.\,m+n\,.\,r+s\,.\,t+u} \pm \frac{x^{\frac{t}{u}+2}}{p+2q\,.\,m+2n\,.\,r+2s\,.\,t+2u} + \frac{x^{\frac{t}{u}+3}}{p+3q\,.\,m+3n\,.\,r+3s\,.\,t+3u} \pm$ &c.

$$(\Sigma)\quad \frac{1}{nqsu} \times : \frac{X^{\pi}}{\pi\,.\,\pi-\omega\,.\,\pi-\delta}\int \frac{x^{\frac{p}{q}}\dot{x}}{1 \mp x} - \frac{1}{\pi\,.\,\pi-\omega\,.\,\pi-\delta}\int \frac{x^{\frac{r}{s}}\dot{x}}{1 \mp x} + \frac{X^{\omega}}{\omega\,.\,\pi-\omega\,.\,\delta-\omega}\int \frac{x^{\frac{r}{s}}\dot{x}}{1 \mp x} - \frac{1}{\omega\,.\,\pi-\omega\,.\,\delta-\omega}\int \frac{x^{\frac{t}{u}}\dot{x}}{1 \mp x} - \frac{X^{\delta}}{\delta\,.\,\delta-\omega\,.\,\pi-\delta}\int \frac{x^{\frac{m}{n}}\dot{x}}{1 \mp x} - \frac{1}{\delta\,.\,\delta-\omega\,.\,\pi-\delta}\int \frac{x^{\frac{t}{u}}\dot{x}}{1 \mp x}.$$

When $\pi\left(\frac{t}{u} - \frac{p}{q}\right)$, $\omega\left(\frac{t}{u} - \frac{r}{s}\right)$, $\delta\left(\frac{t}{u} - \frac{m}{n}\right)$ are poſitive whole numbers,

bers, and X is taken of ſuch value as correſponds with the equation

$$\frac{X^{\pi}}{\pi.\pi-\omega.\pi-\delta}+\frac{X^{\omega}}{\omega.\pi-\omega.\delta-\omega}-\frac{X^{\delta}}{\delta.\delta-\omega.\pi-\delta}=\frac{1}{\pi.\pi-\omega.\pi-\delta}+\frac{1}{\omega.\pi-\omega.\delta-\omega}-\frac{1}{\delta.\delta-\omega.\pi-\delta},$$

the expreſſion will be algebraical; retaining the upper ſigns. If the under ſigns be taken, π, ω, δ muſt be *even* poſitive numbers. See theorems p. 62, 63. when x is $= 1$.

In particular caſes,

2. If ω be a whole poſitive number, $\pi-\delta$ a whole number, the ſigns of the ſeries being affirmative; or *even* numbers, the ſigns being alternately + and −; and the relation of the factors in the propoſed ſeries be ſuch that $\pi.\overline{\pi-\omega}=\delta.\overline{\delta-\omega}$, the expreſſion will be algebraical though π and δ be *fractions* in the former caſe, or *uneven* numbers in the latter; provided x be ultimately taken of ſuch values as ariſe from the above equation. In theſe circumſtances the formula (R) p. 63. becomes

$$(\Sigma)\quad \begin{cases} \frac{1}{n\,q\,u.\omega.\pi-\omega.\delta-\omega}\times:\frac{1}{r+s}\pm\frac{1}{r+2s}+\frac{1}{r+3s}\pm(\omega) \\ +\frac{1}{n\,s\,u.\pi.\pi-\omega.\pi-\delta}\times:\frac{1}{p+q}\pm\frac{1}{p+2q}+\frac{1}{p+3q}\pm(\pi-\delta). \end{cases}$$

3. If π be a whole poſitive number, $\omega-\delta$ a whole number, the ſigns being affirmative; or even numbers, the ſigns being + and −; and $\pi.\overline{\pi-\omega}=\delta.\overline{\pi-\delta}$, the ſum will be expreſſed algebraically though ω and δ be fractions in the former caſe, or *uneven* numbers in the latter, if the ultimate values of x be ſuch as correſpond with the above equation. Under theſe limitations the above-mentioned formula becomes

$$(\Sigma)\quad \begin{cases} \frac{1}{n\,s\,u.\pi.\pi-\omega.\pi-\delta}\times:\frac{1}{p+q}\pm\frac{1}{p+2q}+\frac{1}{p+3q}\pm(\pi) \\ +\frac{1}{n\,q\,u.\omega.\pi-\omega.\delta-\omega}\times:\frac{1}{r+s}\pm\frac{1}{r+2s}+\frac{1}{r+3s}\pm(\omega-\delta). \end{cases}$$

266. (F) $$\frac{a+b.x^{\frac{m}{n}+1}}{p+q.m+n}\pm\frac{a+2b.x^{\frac{m}{n}+2}}{p+2q.m+2n}+\frac{a+3b.x^{\frac{m}{n}+3}}{p+3q.m+3n}\pm\&c.$$

$$(\Sigma)\quad \frac{b}{q\,n\,\pi}\times:{}_{x}X^{\pi}\int\frac{x^{\frac{p}{q}}\dot{x}}{1\mp x}+\frac{}{x-\pi}.\int\frac{x^{\frac{m}{n}}\dot{x}}{1\mp x}.$$

1. When $\pi\left(\frac{m}{n}-\frac{p}{q}\right)$ is a poſitive whole number, the ſigns of the ſeries being affirmative; or an *even* poſitive number, the ſigns being + and −, the expreſſion

will be algebraical, provided X be any value in the equation $X^{\pi} = \frac{x-\pi}{x}$, where x is $= \frac{a}{b} - \frac{p}{q}$. See theorem p. 68.

2. If $x-\pi$ be a negative quantity, and π an *uneven* positive number, and the value of X ſuch as may correſpond with the equation $X^{\pi} = \frac{\pi-x}{x}$, the ſum will be algebraical when the ſigns of the ſeries are alternately + and −. See theorems p. 72. when x is $= 1$.

267. (F) $\frac{a+b.x^{\frac{r}{s}+1}}{p+q.m+n.r+s} \pm \frac{a+2b.x^{\frac{r}{s}+2}}{p+2q.m+2n.r+2s} + \frac{a+3b.x^{\frac{r}{s}+3}}{p+3q.m+3n.r+3s} \pm$ &c.

$$(\Sigma)\quad \frac{b}{q.n.s.\pi} \times \left\{ \frac{x.X^{\delta}}{\delta} \int \frac{x^{\frac{p}{q}}\dot{x}}{1 \mp x} - \frac{x}{\delta} \int \frac{x^{\frac{r}{s}}\dot{x}}{1 \mp x} + \frac{\overline{\pi-x}.X^{\delta-\pi}}{\delta-\pi} \int \frac{x^{\frac{m}{n}}\dot{x}}{1 \mp x} - \frac{\pi-x}{\delta-\pi} \int \frac{x^{\frac{r}{s}}\dot{x}}{1 \mp x} \right\}.$$

1. The ſum will be algebraical when $\pi\left(\frac{m}{n} - \frac{p}{q}\right)$, and $\delta\left(\frac{r}{s} - \frac{p}{q}\right)$, are affirmative whole numbers, and the value of X is ſuch as correſponds with the equation $\frac{x.X^{\delta}}{\delta} + \frac{\overline{\pi-x}.X^{\delta-\pi}}{\delta-\pi} = \frac{x}{\delta} + \frac{\pi-x}{\delta-\pi}$. See theorems p. 82. when x is $= 1$.

Particular caſe.

2. If π be a poſitive whole number, $x = \delta$, and $\pi-x$ or $\delta-\pi$ be negative, though δ and (conſequently) $\delta-\pi$ be fractions, the ſum will be had algebraically when x is any value in the ſame equation, that is in $X^{\pi} = 1$, the ſigns of the ſeries being affirmative. Alſo, if π be an *even* number, the ſum will be had in algebraic terms when the ſigns are + and −, though δ and (conſequently) $\delta-\pi$ be *odd* numbers, the reſt being as before. But it may be obſerved, that the ſeries do in ſuch caſe become eſſentially of the form in N° 263. the terms being multiplied by the conſtant quantity $\frac{a+b}{r+s}$.

268. (F) $\frac{a+b.c+e.x^{\frac{r}{s}+1}}{p+q.m+n.r+s} \pm \frac{a+2b.c+2e.x^{\frac{r}{s}+2}}{p+2q.m+2n.r+2s} + \frac{a+3b.c+3e.x^{\frac{r}{s}+3}}{p+3q.m+3n.r+3s} \pm$ &c.

(Σ)

$$(\Sigma)\qquad \frac{be}{qns}\times\frac{\varkappa\lambda X^{\delta}}{\pi\delta}\int\frac{x^{\frac{p}{q}}\dot{x}}{1\mp x}-\frac{\delta\cdot\varkappa-\delta}{\delta\cdot\delta-\pi}\int\frac{x^{\frac{r}{s}}\dot{x}}{1\mp x}$$
$$-\frac{\varkappa-\pi\cdot\lambda-\pi\cdot X^{\delta-\pi}}{\pi\cdot\delta-\pi}\int\frac{x^{\frac{m}{n}}\dot{x}}{1\mp x}-\frac{\lambda\cdot\varkappa-\delta}{\delta\cdot\delta-\pi}\int\frac{x^{\frac{r}{s}}\dot{x}}{1\mp x}.$$

1. This expreſſion becomes algebraical when δ and $\delta-\pi$ are poſitive whole numbers, the ſigns of the ſeries being affirmative; or *even* numbers, the ſigns being $+$ and $-$; and the ultimate value of x ſuch as ariſes from the equation

$$\frac{\varkappa\lambda X^{\delta}}{\pi\delta}-\frac{\varkappa-\pi\cdot\lambda-\pi\cdot X^{\delta-\pi}}{\pi\cdot\delta-\pi}=\frac{\delta-\varkappa\cdot\lambda-\delta}{\pi\cdot\delta-\pi}.$$

2. Hence, in particular caſes, if we aſſume $\delta=\lambda$, then muſt π be a poſitive whole number, and x a value ariſing from the equation $X^{\pi}=\frac{\varkappa-\pi}{\varkappa}$.

If we take $\pi=\varkappa$, then muſt δ be an affirm. whole number, and x a value in the equation $X^{\delta}=\frac{\lambda-\delta}{\lambda}$.

If π be $=\lambda$, δ muſt be a poſitive whole number, and x a value in the equation $X^{\delta}=\frac{\varkappa-\delta}{\varkappa}$.

If $\varkappa=\delta$, π muſt be a poſitive whole number, and x correſpond with the equation $X^{\pi}=\frac{\lambda-\pi}{\lambda}$.

If π, δ, be *even* numbers, the algebraic ſum will be had when the ſigns of the ſeries are $+$ and $-$. In all theſe caſes, X muſt evidently be leſs than 1.

3. But when the ſigns of the ſeries are alternately $+$ and $-$; and $\varkappa-\pi$, or $\lambda-\pi$ negative, as alſo $\varkappa-\delta$ or $\lambda-\delta$; then muſt δ and $\delta-\pi$ be *odd* poſitive numbers, and x a value in the above equation.

4. Hence, in the firſt particular caſe above, if we take $\pi=2\varkappa$, the expreſſion will be algebraical, whenever π is any *odd* poſitive number, and x a value in the equation $X^{\pi}=1$ (See theorem p. 97. when x is $=1$.). The ſame holds good if we take $\varkappa=\delta$, and $\pi=2\lambda$. In the ſecond caſe, if we make $\delta=2\lambda$, the expreſſion is reducible into finite terms, whenever δ is any *odd* poſitive number, and x a value in the equation $X^{\delta}=1$ (See theorem p. 97. when x is $=1$.). Alſo if $\pi=\lambda$, and $\delta=2\varkappa$, the ſum will be expreſſed in finite terms under the ſame reſtrictions. In theſe caſes one value of X will evidently be always $=1$. It is alſo obſervable here,

that

that the ſeries do under the above limitations become *eſſentially* of the form in N° 266. being multiplied by ſome conſtant quantity.

269. (F) $\dfrac{a+b.c+e.x^{\frac{t}{u}+1}}{p+q.m+n.r+s.t+u} \pm \dfrac{a+2b.c+2e.x^{\frac{t}{u}+2}}{p+2q.m+2n.r+2s.t+2u} + \dfrac{a+3b.c+3e.x^{\frac{t}{u}+3}}{p+3q.m+3n.r+3s.t+3u} \pm$ &c.

$$(\Sigma)\ \frac{be}{qnsu}\times:\frac{\varkappa\lambda X^{\omega}}{\pi\delta\omega}\int\frac{x^{\frac{p}{q}}\dot{x}}{1\mp x}-\frac{\varkappa\lambda}{\pi\delta\omega}\int\frac{x^{\frac{t}{u}}\dot{x}}{1\mp x}+\frac{\pi-\varkappa.\lambda-\pi.X^{\omega-\pi}}{\pi.\delta-\pi.\omega-\pi}\int\frac{x^{\frac{m}{\pi}}\dot{x}}{1\mp x}-$$
$$\frac{\pi-\varkappa.\lambda-\pi}{\pi.\delta-\pi.\omega-\pi}\int\frac{x^{\frac{t}{u}}\dot{x}}{1\mp x}+\frac{\varkappa-\delta.\lambda-\delta.X^{\omega-\delta}}{\delta.\delta-\pi.\omega-\delta}\int\frac{x^{\frac{r}{s}}\dot{x}}{1\mp x}-\frac{\omega-\delta.\lambda-\delta}{\delta.\delta-\pi.\omega-\delta}\int\frac{x^{\frac{t}{u}}\dot{x}}{1\mp x}.$$

1. This expreſſion is algebraical whenever ω, π, δ are poſitive whole numbers, and X any value ariſing from the equation

$$\frac{\varkappa\lambda X^{\omega}}{\pi\delta\omega}+\frac{\pi-\varkappa.\lambda-\pi.X^{\omega-\pi}}{\pi.\delta-\pi.\omega-\pi}+\frac{\varkappa-\delta.\lambda-\delta.X^{\omega-\delta}}{\delta.\delta-\pi.\omega-\delta}=\left(\frac{\varkappa\lambda}{\pi\delta\omega}+\frac{\pi-\varkappa.\lambda-\pi}{\pi.\delta-\pi.\omega-\pi}+\frac{\varkappa-\delta.\lambda-\delta}{\delta.\delta-\pi.\omega-\delta}=\right)\frac{\varkappa-\omega.\lambda-\omega}{\omega.\pi-\omega.\delta-\omega},$$

the ſigns of the ſeries being affirmative. When the ſigns of the ſeries are alternately + and −, ω, π, δ muſt be *even* numbers. See theorems p. 107. when x is $= 1$.

In particular caſes,

2. If the value of $\frac{Q}{\omega-\pi}$ be negative (ſee p. 106.), and equal to $\frac{P}{\omega}$, and π be a whole poſitive number, $\omega-\delta$ a whole number, the algebraic ſum is attainable, though ω and δ be fractions, provided the ultimate value of x be taken ſuch as may correſpond with the above equation, that is, with $\frac{\varkappa\lambda X^{\omega}}{\pi\delta\omega}=\frac{\varkappa-\pi.\lambda-\pi.X^{\omega-\pi}}{\pi.\delta-\pi.\omega-\pi}$; in which, if we ſuppoſe $X=1$, we ſhall have $\varkappa=\delta$, and $\omega=\lambda$, or $\varkappa=\omega$, and $\delta=\lambda$. Under theſe limitations the general formula p. 107. becomes

$$(\Sigma)\ \begin{cases}\frac{Pq}{u\omega}\times:\frac{1}{p+q}\pm\frac{1}{p+2q}+\frac{1}{p+3q}\pm(\pi)\\ +\frac{Rs}{u.\omega-\delta}\times:\frac{1}{r+s}\pm\frac{1}{r+2s}+\frac{1}{r+3s}\pm(\omega-\delta).\end{cases}$$

3. If $\frac{R}{\omega-\delta}$ be negative, and $=\frac{P}{\omega}$; and δ be a whole poſitive number, $\omega-\pi$ a whole number; the ſum is expreſſed algebraically though ω and π are fractions. And ſuppoſing

ſuppoſing the ultimate value of x to be 1 in the above equation, that is, taking $\varkappa = \pi$ and $\lambda = \omega$, or $\varkappa = \omega$ and $\lambda = \pi$, the ſame expreſſion becomes

$$(\Sigma)\quad \begin{cases} \frac{Pq}{u\omega} \times : \frac{1}{p+q} \pm \frac{1}{p+2q} + \frac{1}{p+3q} \pm (\delta) \\ \mp \frac{Qn}{u \cdot \omega - \pi} \times : \frac{1}{m+n} \pm \frac{1}{m+2n} + \frac{1}{m+3n} \quad (\omega - \pi). \end{cases}$$

4. The comparing of $\frac{Q}{\omega - \pi}$ and $\frac{R}{\omega - \delta}$ requires $\pm \pi \mp \delta$ to be a whole number, and when x is ſuppoſed to become 1, it is requiſite that π be $= \delta$; in this caſe therefore the ſum is infinite when x attains that value.

270. (F) $$\frac{a+b \cdot c+e \cdot f+g\,(k)\, x^{\frac{w}{v}+1}}{p+q \cdot m+n \cdot r+s \cdot t+u\,(l+1)} \pm \frac{a+2b \cdot c+2e \cdot f+2g\,(k)\, x^{\frac{w}{v}+2}}{p+2q \cdot m+2n \cdot r+2s \cdot t+2u\,(l+1)}$$

$$\mp \frac{a+3b \cdot c+3e \cdot f+3g\,(k)\, x^{\frac{w}{v}+3}}{p+3q \cdot m+3n \cdot r+3s \cdot t+3u\,(l+1)} \pm \&c.$$

$$= \begin{cases} X^{\frac{w}{v}} \times AX^{-\frac{p}{q}} + BX^{-\frac{m}{n}} + CX^{-\frac{r}{s}} + DX^{-\frac{t}{u}}\,(l+1) \times \int \frac{x^{\frac{p}{q}}\,\dot{x}}{1-x} \\ (\Sigma) \quad qX^{\frac{w}{v}+1} \times \begin{cases} -BX^{-\pi} \times : \frac{1}{p+q} \pm \frac{X}{p+2q} + \frac{X^2}{p+3q}\,(\pi) \\ -CX^{-\rho} \times : \frac{1}{p+q} \pm \frac{X}{p+2q} + \frac{X^2}{p+3q}\,(\rho) \\ -DX^{-\sigma} \times : \frac{1}{p+q} \pm \frac{X}{p+2q} + \frac{X^2}{p+3q}\,(\sigma) \end{cases} \\ \&c.\ \text{to } l \text{ terms.} \end{cases}$$

$$\left.\begin{matrix} A = \\ B = \\ C = \\ \&c. \end{matrix}\right\} \frac{beg\,(k)}{qnsu\,(l+1)} \times \begin{cases} \frac{\alpha\ \beta\ \gamma}{\pi\ \rho\ \sigma}\ \&c. \\ \frac{\alpha'\ \beta'\ \gamma'}{\pi'\ \rho'\ \sigma'}\ \&c. \\ \frac{\alpha''\ \beta''\ \gamma''}{\pi''\ \rho''\ \sigma''}\ \&c. \end{cases}$$

α, β, γ, &c. $= \frac{a}{b} - \frac{p}{q}, \frac{c}{e} - \frac{p}{q}, \frac{f}{g} - \frac{p}{q}$, &c. reſpectively.

$\alpha', \beta', \gamma',$

$$\alpha, \beta, \gamma, \text{ \&c.} = \frac{a}{b} - \frac{m}{n}, \text{ \&c.}$$

$$\alpha', \text{ \&c.} = \frac{a}{b} - \frac{r}{s}, \text{ \&c.}$$

$$\pi, \rho, \sigma, \text{ \&c.} = \frac{m}{n} - \frac{p}{q}, \frac{r}{s} - \frac{p}{q}, \frac{t}{u} - \frac{p}{q}, \text{ \&c.}$$

$$\pi', \text{ \&c.} = \frac{p}{q} - \frac{m}{n}, \text{ \&c.}$$

$$\pi'', \text{ \&c.} = \frac{p}{q} - \frac{r}{s}, \text{ \&c.}$$

The index $\frac{w}{v}$ is $= \frac{m}{n}, \frac{r}{s}$, or $\frac{t}{u}$, &c. according as l is 1, 2, or 3, &c. and π, ρ, σ, &c. are positive whole numbers.

271. (F)
$$\frac{a+b\,.\,c+e\,(k)\,x}{p+q\,.\,m+n\,(l+1)} + \frac{a+2b\,.\,c+2e\,(k)\,g\,dx^2}{p+2q\,.\,m+2n\,(l+1)} + \frac{a+3b\,.\,c+3e\,(k)\,g\,.\,\frac{g-1}{2}\,.\,d^2x^3}{p+3q\,.\,m+3n\,(l+1)}$$
$$+ \frac{a+4b\,.\,c+4e\,(k)\,g\,.\,\frac{g-1}{2}\,.\,\frac{g-2}{3}\,.\,d^3x^4}{p+4q\,.\,m+4n\,(l+1)} + \text{ \&c.}$$

$$(\Sigma)\begin{cases} AX^{-\frac{p}{q}} + \frac{p+q\,.\,p+2q\,(\pi)\,BX^{-\frac{m}{n}}}{gq+p+2q\,.\,gq+p+3q\,(\pi)\,d^{\pi}} + \frac{p+q\,.\,p+2q\,(\rho)\,CX^{-\frac{r}{s}}}{gq+p+2q\,.\,gq+p+3q\,(\rho)\,d^{\rho}}(l+1) \times \int \frac{x^{\frac{p}{q}}\,\dot{x}}{\overline{1-dx}^{g}} \\ -X \times \begin{cases} \frac{p+q\,.\,p+2q\,(\pi)\,B\,.\,\overline{1-dX}^{g+1}}{gq+p+2q\,.\,gq+p+3q\,(\pi)\,X^{\pi-1}\,d^{\pi}} \times : \frac{q}{p+q} + \frac{gq+p+2q\,.\,dX}{p+q\,.\,p+2q} + \frac{gq+p+2q\,.\,gp+p+3q\,.\,d^2X^2}{p+q\,.\,p+2q\,.\,p+3q}(\pi) \\ \frac{p+q\,.\,p+2q\,(\rho)\,C\,.\,\overline{1-dX}^{g+1}}{gq+p+2q\,.\,gq+p+3q\,(\rho)\,X^{\rho-1}\,d^{\rho}} \times : \frac{q}{p+q} + \frac{gq+p+2q\,.\,dX}{p+q\,.\,p+2q}(\rho) \\ \text{\&c. to } l \text{ terms.} \end{cases} \end{cases}$$

A, B, C, &c. π, ρ, σ, &c. as in the preceding theorem.

272. (F)
$$\frac{1}{b-e+1\,.\,b+1\,.\,2b-g-2e+2\,.\,2b+g+2} \pm \frac{1}{b-e+2\,.\,b+2\,.\,2b-g-2e+4\,.\,2b+g+4} + \frac{1}{b-e+3\,.\,b+3\,.\,2b-g-2e+6\,.\,2b+g+6} \pm \text{ \&c.}$$

$$(\Sigma)\begin{cases} \frac{1}{g+e\,.\,\frac{1}{2}g^2+ge} \times : \frac{1}{2b-g-2e+2} \pm \frac{1}{2b-g-2e+4} + \frac{1}{2b-g-2e+6}(g+e) \\ -\frac{1}{g^2e+2ge^2} \times : \frac{1}{b-e+1} \pm \frac{1}{b-e+2} + \frac{1}{b-e+3}(e). \end{cases}$$

1. When

1. When the ſigns are affirmative, and g is any odd number (e being any whole number whatever), $\varpi\left(\frac{g+2e}{2}\right)$ and $\delta\left(\frac{g}{2}\right)$ will be *fractions*.

2. When the ſigns are alternate, g any number in the progreſſion 2, 6, 10, 14, 18, &c. and e any number in the progreſſion 2, 4, 6, 8, 10, &c. ϖ and δ will be *odd* numbers. In the former caſe, $\omega\,(g+e)$ will evidently be always a whole number; and in the latter, always an even number.

273. (F) $$\frac{1}{b-e+2\,.\,2b+g+4\,.\,2b+g-2e+4\,.\,b+g+2} \pm \frac{1}{b-e+3.2b+g+6.2b+g-2e+6.b+g+3} + \frac{1}{b-e+4.2b+g+8.2b+g-2e+8.b+g+4} \pm \&c.$$

(Σ) $$\begin{cases} \frac{1}{g+e\,.\,g^2+2ge} \times : \frac{1}{b-e+2} \pm \frac{1}{b-e+3} + \frac{1}{b-e+4} \pm (g+e) \\ -\frac{1}{\frac{1}{2}g^2e+ge^2} \times : \frac{1}{2b+g-2e+4} \pm \frac{1}{2b+g-2e+6} + (e). \end{cases}$$

1. What is obſerved in the laſt theorem is applicable in this; writing ω for ϖ, and ϖ for ω.

2. If, in the preceding formulæ (Nos 265, 269, 272, 273,) the quantities $\varpi-\delta$, $\omega-\delta$, or $\omega-\varpi$, &c. ſhould come out negative, write one value for the other in the theorem, that is for ϖ write the value of δ, and for δ write the value of ϖ, &c. and the algebraic ſum will then be truly exhibited in thoſe caſes.

274. (F) $$\mathrm{A},s\,.\,\frac{a}{r}\sqrt{r^2-b^2} - \frac{b}{r}\sqrt{r^2-a^2} + \mathrm{A},s\,.\,\frac{b}{r}\sqrt{r^2-c^2} - \frac{c}{r}\sqrt{r^2-b^2} + \mathrm{A},s\,.\,\frac{c}{r}\sqrt{r^2-d^2} - \frac{d}{r}\sqrt{r^2-c^2} + \&c.$$

(Σ) A, $s\,.\,a$, radius r.

a being any number not greater than r; and b, c, d, &c. any numbers decreaſing from a to 0.

275. (F) $$\mathrm{A},\sigma.\frac{ab+\sqrt{r^2-a^2.r^2-b^2}}{r} + \mathrm{A},\sigma.\frac{bc+\sqrt{r^2-b^2.r^2-c^2}}{r} + \mathrm{A},\sigma\,.\,\frac{cd+\sqrt{r^2-c^2.r^2-d^2}}{r} + \&c.$$

(Σ) A, $\sigma\,.\,a$, rad. 1.

a, b, c, &c. increaſing from any value to r.

276. (F) $$\mathrm{A},t\,.\,\frac{r^2\,.\,a-b}{r^2+ab} + \mathrm{A},t\,.\,\frac{r^2\,.\,b-c}{r^2+bc} + \mathrm{A},t\,.\,\frac{r^2\,.\,c-d}{r^2+cd} + \&c.$$

(Σ) A, $t\,.\,a$, rad. r.

a, b, c, &c. decreaſing from any value to 0.

277. (F) $$\mathrm{A},\tau\,.\,\frac{r^2+ab}{b-a} + \mathrm{A},\tau\,.\,\frac{r^2+bc}{c-b} + \mathrm{A},\tau\,.\,\frac{r^2+cd}{d-c} + \&c.$$

(Σ) A, τ . a, rad. r.

a, b, c, &c. increasing from any value to infinity.

278. (F) $\mathrm{A}, \int . \frac{ab}{\sqrt{a^2-1.b^2-1}+1} + \mathrm{A}, \int . \frac{bc}{\sqrt{b^2-1.c^2-1}+1} + \mathrm{A}, \int . \frac{cd}{\sqrt{c^2-1.d^2-1}+1} + \&c.$

(Σ) A, $\int$. a, rad. 1.

a, b, c, &c. being decreasing quantities from any value to 1.

279. (F) $\mathrm{A}, s . \frac{ab}{\sqrt{a^2-1}-\sqrt{b^2-1}} + \mathrm{A}, s . \frac{bc}{\sqrt{b^2-1}-\sqrt{c^2-1}} + \mathrm{A}, s . \frac{cd}{\sqrt{c^2-1}-\sqrt{d^2-1}} + \&c.$

(Σ) A, s . a, rad. 1.

a, b, c, &c. increasing from any value not less than 1 to infinity.

280. (F) $\mathrm{A}, v . \frac{a+b-2r.\sqrt{2ar-a^2}.\overline{b-r}-\sqrt{2br-b^2}.\overline{a-r}}{r.\sqrt{2ar-a^2}-r.\sqrt{2br-b^2}} + \mathrm{A}, v . \frac{b+c-2r.\sqrt{2br-b^2}.\overline{c-r}-\sqrt{2cr-c^2}.\overline{b-r}}{r.\sqrt{2br-b^2}-r.\sqrt{2cr-c^2}}$

$+ \mathrm{A}, v . \frac{c+d-2r.\sqrt{2cr-c^2}.\overline{d-r}-\sqrt{2dr-d^2}.\overline{c-r}}{r.\sqrt{2cr-c^2}-r.\sqrt{2dr-d^2}} + \&c.$

(Σ) A, v . a, rad. r.

a, b, c, &c. increasing from r to $2r$.

281. (F) $\mathrm{A}, v . \frac{r.\overline{a-b}.\sqrt{2ra-a^2}.\sqrt{2rb-b^2}}{r-b.\sqrt{2ra-a^2}+r-a.\sqrt{2rb-b^2}} + \mathrm{A}, v . \frac{r.\overline{b-c}.\sqrt{2rb-b^2}.\sqrt{2rc-c^2}}{r-c.\sqrt{2rb-b^2}+r-b.\sqrt{2rc-c^2}}$

$+ \mathrm{A}, v . \frac{r.\overline{c-d}.\sqrt{2rc-c^2}.\sqrt{2rd-d^2}}{r-d.\sqrt{2rc-c^2}+r-c.\sqrt{2rd-d^2}} + \&c.$

(Σ) A, v . a, rad. r.

a, b, c, &c. decreasing from r to 0.

In the last eight formulæ, A denotes circular arc; s, σ, t, τ, f, ς, v, ν, the sine, co-sine, tangent, co-tangent, secant, co-secant, versed sine, and versed sine supplement, respectively.

282. (F) $x + \frac{x^3}{2 \cdot 3r^2} + \frac{3x^5}{2 \cdot 4 \cdot 5r^4} + \frac{3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7r^6} + \&c.$

(Σ) Circ. arc, rad. r, sin. x.

283. (F) $x + \frac{c^2 x^3}{2 \cdot 3a^4} + \frac{\overline{4a^2c^2-c^4} \cdot x^5}{2 \cdot 4 \cdot 5a^8} + \frac{\overline{8a^4c^4-4a^2c^4+c^6} \cdot 3x^7}{2 \cdot 4 \cdot 6 \cdot 7a^{12}} + \&c.$

(Σ) Elliptical arc, a the semitransv. c the semiconj. x the distance of an ordinate (parallel to the conjugate) from the center.

284. (F) $x \times : 1 + \frac{q^2}{2 \cdot 3} - \frac{q^4}{2 \cdot 4 \cdot 5} + \frac{3q^6}{2 \cdot 4 \cdot 6 \cdot 7} - \frac{3 \cdot 5q^8}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} + \&c.$

(Σ) Parabolic arc; x an ordinate, $q = \frac{x}{\frac{1}{2}\text{par.}}$ (less than 1).

285. (F)

285. (F) $c \times : A + \frac{q}{2} B - \frac{q^2}{2.4} C + \frac{3q^3}{2.4.6} D - \frac{3.5q^4}{2.4.6.8} +$ &c.

(Σ) Hyperbolic arc (beginning at the vertex); a = the ſemitranſv. axe, c = the ſemiconj. $q = \frac{a^2+c^2}{c^4}$, y an ordinate, $A = L . \frac{y+x}{c}$,

$$B = \frac{yx - c^2 A}{2}, C = \frac{y^3 x - 3c^2 B}{4}, D = \frac{y^5 x - 5c^2 C}{6}, E = \frac{y^7 x - 7c^2 D}{8}, \text{ &c.}$$

where $x = \sqrt{c^2+y^2}$.

Here we might introduce all the variety of ſeries which expreſs the areas of the conic ſections, circular ſegments, &c. as alſo thoſe ſeries which exhibit the ſurfaces and ſolidities of the ſphere, ſpheroid, paraboloid, &c. but as theſe may be met with in various authors, I ſhall omit them. Several ſeries of this kind are given in Dr. Hutton's excellent Treatiſe on Menſuration.

286. (F) $\frac{x^{n+2}}{n+2} + \frac{x^{m+n+2}}{m+1.m+n+2} + \frac{x^{2m+n+2}}{2m+1.2m+n+2} + \frac{x^{3m+n+2}}{3m+1.3m+n+2} +$ &c.

(Σ) $\frac{x^{n+1}-1}{n+1} \int \frac{\dot{x}}{1-x^m} + \frac{1}{n+1} \times : \frac{x^{n+2-m}}{n+2-m} + \frac{x^{n+2-2m}}{n+2-2m} + \frac{x^{n+2-3m}}{n+2-3m} +$ &c. (x)

Where $\frac{n+1}{m}$ muſt be an affirm. integer.

287. (F) $\frac{n}{m} + \frac{n.n+r}{m.m+r} + \frac{n.n+r.n+2r}{m.m+r.m+2r} +$ &c.

(Σ) $\frac{n}{m-n-r}$.

288. (F) $a + \frac{an}{m} + \frac{an.n-r}{m.m+r} + \frac{an.n+r.n+2r}{m.m+r.m+2r} +$ &c.

(Σ) $\frac{a.m-r}{m-n-r}$.

289. (F) $\overline{m \pm n}^r + \overline{m \pm 2n}^r + \overline{m \pm 3n}^r + \overline{m \pm 4n}^r +$ &c. $\overline{m \pm zn}^r$.

(S) $zm^r \pm Arm^{r-1}n + \frac{B}{2} r.r-1.m^{r-2}n^2 \pm \frac{C}{2.3} r.r-1.r-2.m^{r-3}n^3 +$

$\frac{D}{2.3.4} r.r-1.r-2.r-3.m^{r-4}n^4 \pm$ &c. till it terminates.

A, B, C, &c. are the ſums of the ſeries in Order III. Nos 6, 7, 8, &c.

290. (F) $x^{b+1} \times : \frac{a}{b+c} + \frac{c-1}{b+d} A + \frac{d-1}{b+e} B + \frac{e-1}{b+f} C + \frac{f-1}{b+g} D +$ &c.

(Σ) $\int ax^b \dot{x} = \frac{ax^{b+1}}{b+1}$.

291. (F) $\frac{x^2-1}{x} - \frac{x^4-1}{2x^2} + \frac{x^6-1}{3x^3} - \frac{x^8-1}{4x^4} + \frac{x^{10}-1}{5x^5} - \&c.$

(Σ) $\int x^{-1}\dot{x} = \mathrm{L}.x.$

292. (F) $\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \frac{1}{5x^5} - \&c.$

(Σ) $\mathrm{L}.\frac{1+x}{x}.$

293. (F) $\frac{}{a+bx+cx^2+dx^3+ex^4+\&c.}^{n}$

$$(\Sigma)\begin{cases} a^n + \frac{nbA}{a}x + \frac{2ncA+\overline{n-1}.bB}{2a}x^2 + \frac{3ndA+\overline{2n-1}.cB+\overline{n-2}.bC}{3a}x^3 \\ + \frac{4neA+\overline{3n-1}.dB+\overline{2n-2}.2C+\overline{n-3}.bD}{4a}x^4 + \frac{5nfA+\overline{4n-1}.eB+\overline{3n-2}.dC+\overline{2n-3}.cD+\overline{n-4}.bE}{5a}x^5 \\ + \frac{6ngA+\overline{5n-1}.fB+\overline{4n-2}.eC+\overline{3n-3}.dD+\overline{2n-4}.cE+\overline{n-5}.bF}{6a}x^6 + \&c. \end{cases}$$

A, B, C, D, &c. being the coefficients of the terms immediately preceding thoſe wherein they firſt ariſe.

By means of this formula may the roots of any adfected equation be exhibited by ſeries; and conſequently when ſuch roots can be had in finite terms, the ſums of thoſe infinite ſeries will be obtained. And from this formula will a great variety of infinite ſeries ariſe whoſe ſums will be had in finite expreſſions by taking 2, 3, 4, &c. terms of (F), and aſſuming different values for a, b, c, &c. and n. From whence, by addition and ſubtraction, we alſo derive a number of ſeries of a different claſs, with their ſummations; among which are exhibited the roots of certain adfected equations, of any dimenſion.

294. (F) $2\sqrt[n]{b} \times :1 \pm \frac{1-n.c^2}{2n^2b^2} + \frac{1-n.1-2n.1-3n.c^4}{2\cdot3\cdot4n^4b^4} \pm \frac{1-n.1-2n.1-3n.1-4n.1-5n.c^6}{2\cdot3\cdot4\cdot5\cdot6n^6b^6} + \&c.$

(Σ) $\sqrt[n]{b+\sqrt{\pm c^2}} + \sqrt[n]{b-\sqrt{\pm c^2}}$; being roots of the general equation

$$x^n \mp n r^{\frac{1}{n}} x^{n-2} \mp \frac{n.n-3.r^{\frac{2}{n}}}{2} x^{n-4} \mp \frac{n.n-4.n-5.r^{\frac{3}{n}}}{2\cdot3} x^{n-6} \mp \frac{n.n-5.n-6.n-7.r^{\frac{4}{n}}}{2\cdot3\cdot4} x^{n-8} \mp \&c. - 2b = 0.$$

Where r $(= b^2 - c^2)$ muſt be a poſitive quantity, or 0.

295. (F)

295. (F) $\frac{2b}{\frac{n-1}{c^n}} \times : \frac{1}{n} + \frac{1-n.1-2n.b^2}{2.3n^2c^2} + \frac{1-n.1-2n.1-3n.1-4n.b^4}{2.3.4.5.n^3c^4} +$

$$\frac{1-n.1-2n.1-3n.1-4n.1-5n.1-6n.b^6}{2.3.4.5.6.7n^7c^6} + \text{&c.}$$

(Σ) $\sqrt{\sqrt{\pm c^2}+b} - \sqrt{\sqrt{\pm c^2}-b}$; being roots of the above equation, when the signs are positive, and $r\ (=c^2-b^2)$ is a positive quantity, or 0; n being an odd number.

Hence arise the following series,

296. (F) $2\sqrt{b} \times : 1 - \frac{c^2}{2.4b^2} - \frac{3.5c^4}{2.4.6.8b^4} - \frac{3.5.7.9c^6}{2.4.6.8.10.12b^6} - \text{&c.}$

(Σ) $\sqrt{b+c} + \sqrt{b-c}$; being a root of the equat. $x^2 - 2\sqrt{b^2-c^2} - 2b = 0$.

297. (F) $2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3.6b^2} - \frac{2.5.8c^4}{3.6.9.12b^4} - \frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6} - \text{&c.}$

(Σ) $\sqrt[3]{b+c} + \sqrt[3]{b-c}$; being a root of the equat. $x^3 - 3\sqrt[3]{b^2-c^2}.x = 2b$.

298. (F) $2\sqrt[4]{b} \times : 1 - \frac{3c^2}{4.8b^2} - \frac{3.7.11c^4}{4.8.12.16b^4} - \frac{3.7.11.15.19c^6}{4.8.12.16.20.24b^6} - \text{&c.}$

(Σ) $\sqrt[4]{b+c} + \sqrt[4]{b-c}$; being a root of the equat. $x^4 - 4\sqrt[4]{b^2-c^2}.x^2 - 2\sqrt{b^2-c^2} - 2b = 0$.

299. (F) $2\sqrt[5]{b} \times : 1 - \frac{4c^2}{5.10b^2} - \frac{4.9.14c^4}{5.10.15.20b^4} - \frac{4.9.14.19.24c^6}{5.10.15.20.25.30b^6} - \text{&c.}$

(Σ) $\sqrt[5]{b+c} + \sqrt[5]{b-c}$; a root of the equat. $x^5 - 5\sqrt[5]{b^2-c^2}.x^3 - 5.\sqrt[5]{(b^2-c^2)^2}.x - 2b = 0$.

300. (F) $2\sqrt[7]{b} \times : 1 - \frac{6c^2}{7.14b^2} - \frac{6.13.20c^4}{7.14.21.28b^4} - \frac{6.13.20.27.36c^6}{7.14.21.28.35.42b^6} - \text{&c.}$

(Σ) $\sqrt[7]{b+c} + \sqrt[7]{b-c}$; a root of the equat. $x^7 - 7\sqrt[7]{b^2-c^2}.x^5 - 14.\sqrt[7]{(b^2-c^2)^2}.x^3 - 7.\sqrt[7]{(b^2-c^2)^3}.x = b$.

&c.

301. (F) $\frac{2b}{c^{\frac{2}{3}}} \times : \frac{1}{3} + \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11b^4}{3.6.9.12.15.c^4} + \text{&c.}$

(Σ) $\sqrt[3]{c+b} - \sqrt[3]{c-b}$.

302. (F) $\frac{2b}{c^{\frac{4}{5}}} \times : \frac{1}{5} + \frac{4.9b^2}{5.10.15c^2} + \frac{4.9.14.19b^4}{5.10.15.20.25c^4} + \text{&c.}$

(Σ) $\sqrt[5]{c+b} - \sqrt[5]{c-b}$.

&c.

If

If in the general formula N° 293. we take $b = \frac{1}{2^n}$, and $c = \frac{x}{2^n}$, we obtain

303. (F) $1 \pm \frac{1-n.x^2}{2n^2} + \frac{1-n.1-2n.1-3n.x^4}{2.3.4n^4} \pm \frac{1-n.1-2n.1-3n.1-4n.1-5n.x^6}{2.3.4.5.6n^6} + \&c.$

(Σ) $\frac{1}{2}\sqrt[n]{1+\sqrt{\pm x^2}} + \frac{1}{2}\sqrt[n]{1-\sqrt{\pm x^2}}.$

And by making $b = \frac{x}{2x^n}$, and $c = \frac{1}{2x^n}$, the formula N° 294. becomes

304. (F) $\frac{1}{n} \pm \frac{1-n.1-2n.x^2}{2.3n^3} + \frac{1-n.1-2n.1-3n.1-4n.x^4}{2.3.4.5n^5} \pm \&c.$

(Σ) $\frac{\sqrt[n]{1+\sqrt{\pm x^2}} - \sqrt[n]{1-\sqrt{\pm x^2}}}{2x}.$

Hence arise the two following formulæ.

305. (F) $1 + \frac{1-n.1-2n.1-3n.x^4}{1.2.3.4n^4} + \frac{1-n.1-2n.1-3n.1-4n.1-5n.1-6n.1-7n.x^8}{1.2.3.4.5.6.7.8n^6} + \&c.$

(Σ) $\frac{1}{2} \times : \sqrt[n]{1+x} + \sqrt[n]{1-x} + \sqrt[n]{1+\sqrt{-x^2}} + \sqrt[n]{1-\sqrt{-x^2}}.$

306. (F) $\frac{1}{n} + \frac{1-n.1-2n.1-3n.1-4n.x^4}{1.2.3.4.5n^5} + \frac{1-n.1-2n.1-3n.1-4n.1-5n.1-6n.1-7n.1-8n.x^8}{1.2.3.4.5.6.7.8.9n^9} + \&c.$

(Σ) $\frac{1}{2x} \times : \sqrt[n]{1+x} - \sqrt[n]{1-x} + \sqrt[n]{1+\sqrt{-x^2}} - \sqrt[n]{1-\sqrt{-x^2}}.$

See Dr. Hutton's Paper on Cubic Equations and Infinite Series, Phil. Tranſ. Vol. LXX. Art. 25.

307. (F) $\frac{d}{b} - \frac{1.2^2d^2}{4b^3} + \frac{1.3.2^4d^3}{4.6b^5} - \frac{1.3.5.2^6d^4}{4.6.8b^7} + \frac{1.3.5.7.2^8d^5}{4.6.8.10b^9} - \&c.$

(Σ) The affirmative root of the equation $x^2 + bx = d$.

308. (F) $b - \frac{d}{b} - \frac{1.2^2d^2}{4b^3} - \frac{1.3.2^4d^3}{4.6b^5} - \frac{1.3.5.2^6d^4}{4.6.8b^7} - \&c.$

(Σ) The greater root of the equation $x^2 - bx = -d$.

309. (F) $2b \times : 1 - \frac{1.2^2d^2}{4b^4} - \frac{1.3.5.2^6d^4}{4.6.8b^8} - \frac{1.3.5.7.9.2^{10}d^6}{4.6.8.10.12b^{12}} - \&c.$

(Σ) $\sqrt{\frac{b^2}{4}+d} + \sqrt{\frac{b^2}{4}-d}.$

310. (F) $2b \times : \frac{d}{b^2} + \frac{1.3.2^4d^3}{4.6b^6} + \frac{1.3.5.7.2^8d^5}{4.6.8.10b^{10}} + \frac{1.3.5.7.9.11.2^{12}d^7}{4.6.8.10.12.14b^{14}} + \&c.$

(Σ) $\sqrt{\frac{b^2}{4}+d} - \sqrt{\frac{b^2}{4}-d}.$

311. (F)

311. (F) $\sqrt{a} \times : 1 \pm \frac{1}{2} \cdot \frac{b \pm c}{a} - \frac{1}{2.2^2} \cdot \frac{b^2 \pm 2bc + c^2}{a^2} \pm \frac{3}{2.3.2^3} \cdot \frac{b^3 \pm 3b^2c + 3bc^2 \pm c^3}{a^3} -$

$$\frac{3 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 2^4} \cdot \frac{b^4 \pm 4b^3c + 6b^2c^2 \pm 5bc^3 + c^4}{a^4} \pm \&c.$$

(Σ) $\overline{a \pm b \pm c}\Big|^{\frac{1}{2}}$.

312. (F) $\sqrt[3]{a} \times : 1 \pm \frac{1}{3} \cdot \frac{b \pm c}{a} - \frac{2}{2.3^2} \cdot \frac{b^2 \pm 2bc + c^2}{a^2} \pm \frac{2 \cdot 5}{2.3.3^3} \cdot \frac{b^3 \pm 3b^2c + 3bc^2 \pm c^3}{a^3} - \&c.$

(Σ) $\overline{a \pm b \pm c}\Big|^{\frac{1}{3}}$.

&c.

313. (F) $1 + \frac{m \cdot m+n \cdot x^2 a}{n \cdot 2n} + \frac{m+2n \cdot m+3n \cdot x^2 b}{3n \cdot 4n} + \frac{m+4n \cdot m+5n \cdot x^2 c}{5n \cdot 6n} + \&c.$

(Σ) $\frac{1}{\overline{1+x}\Big|^{\frac{m}{n}}} + \frac{1}{\overline{1-x}\Big|^{\frac{m}{n}}}$.

Where a, b, c, &c. are the preceding terms.

By means of the following table of fluxions and fluents may a great variety of infinite ſeries with their ſummations be exhibited, by expanding the fluxionary expreſſions, and taking the fluents. An example or two will explain it.

Form IV. $\int \frac{1}{1-x^2} \times \dot{x}$, by diviſion, &c. becomes

314. (F) $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \&c. = (\Sigma)\ \frac{1}{2}\,\mathrm{L} \cdot \frac{1+x}{1-x}$.

Form VI. $\int \frac{1}{1-x^4} \times \dot{x} =$

315. (F) $x + \frac{x^5}{5} + \frac{x^9}{9} + \frac{x^{13}}{13} + \&c. = (\Sigma)\ \frac{1}{4}\,\mathrm{L} \cdot \frac{1+x}{1-x} + \frac{1}{2}\,\mathrm{A},\ t \cdot x$.

Form XIX. $\int \frac{1}{1+x^4} \times x\dot{x} =$

316. (F) $\frac{x^2}{2} - \frac{x^6}{6} + \frac{x^{10}}{10} - \frac{x^{14}}{14} + \&c. = (\Sigma)\ \frac{1}{2}\,\mathrm{A},\ t \cdot x^2$.

&c.

And by the addition or ſubtraction of ſuch of theſe expreſſions, the terms of whoſe reſulting ſeries may be ſo connected that the denominators (or exponents) may form one continued arith. progreſſion, we derive a number of other ſeries, and their ſummations.

$\int \frac{\dot{x}}{1+x^4}$

$$\int \frac{\dot{x}}{1+x^4} - \int \frac{x^2\dot{x}}{1+x^4} =$$

317. (F) $x - \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7}$ &c. $= (\Sigma)\sqrt{\frac{1}{8}} \times \mathrm{L}.\ \frac{1+2^{\frac{1}{2}}x+x^2}{1-2^{\frac{1}{2}}x+x^2}$.

$$\int \frac{\dot{x}}{1-x^6} + \int \frac{x^4\dot{x}}{1-x^6} =$$

318. (F) $x + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^{11}}{11}$ &c. $= (\Sigma)\frac{1}{2}\mathrm{L}.\ \frac{1+x+x^2}{1-x+x^2} + \frac{1}{3}\mathrm{L}.\ \frac{1+x^3}{1-x^3}$.

&c.

From hence alſo may a variety of ſeries of a different claſs be obtained, by multiplying the above ſeries and their reſpective ſums by ſome fluxionary expreſſion, and taking the fluents. And by repeating the operation, with the ſame or ſome other fluxionary quantity, different ſeries and expreſſions for their ſums will ariſe.

$$\int \frac{\dot{x}}{1+x} =$$

319. (F) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} -$ &c. $= (\Sigma)\ \mathrm{L}.\ \overline{1+x}$.

Multiplying this ſeries by $x^{-1}\dot{x}$, and taking the fluent, we get

320. (F) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{2^2} -$ &c. $= (\Sigma)\int x^{-1}\dot{x}\int \frac{\dot{x}}{1+x}$.

Multiply again by $\dot{x}$, and take the fluent, and there ariſes

321. (F) $\frac{x^2}{1^2\cdot 2} - \frac{x^3}{2^2\cdot 3} + \frac{x^4}{3^2\cdot 4} - \frac{x^5}{4^2\cdot 5} +$ &c. $= (\Sigma)\int \dot{x}\int x^{-1}\dot{x}\int \frac{\dot{x}}{1+x}$:

This ſeries multiplied by $x^{-1}\dot{x}$, and the fluent taken, the reſult is

322. (F) $\frac{x^2}{1\cdot 2^2} - \frac{x^3}{2\cdot 3^2} + \frac{x^4}{3\cdot 4^2} - \frac{x^5}{4\cdot 5^2} +$ &c. $= (\Sigma)\int x^{-1}\dot{x}\int \dot{x}\int x^{-1}\dot{x}\int \frac{\dot{x}}{1+x}$.

&c.

By the ſame proceſs we find

323. (F) $\frac{x^3}{1\cdot 3^2} - \frac{x^4}{2\cdot 4^2} + \frac{x^5}{3\cdot 5^2} - \frac{x^6}{4\cdot 6^2} +$ &c. $= (\Sigma)\int x^{-1}\dot{x}\int x\dot{x}\int x^{-1}\dot{x}\int \frac{\dot{x}}{1+x}$.

324. (F) $\frac{x^5}{1\cdot 3\cdot 5^2} - \frac{x^6}{2\cdot 4\cdot 6^2} + \frac{x^7}{3\cdot 5\cdot 7^2} -$ &c. $= (\Sigma)\int x^{-1}\dot{x}\int x\dot{x}\int x^{-1}\dot{x}\int x\dot{x}\int x^{-1}\dot{x}\int \frac{\dot{x}}{1+x}$.

&c.

325. (F) $$x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \&c. = (\Sigma)\sqrt{\int x^{-1}\dot{x}\int\frac{\dot{x}}{1+x^2}}.$$

326. (F) $$\frac{x^2}{2^2} - \frac{x^6}{6^2} + \frac{x^{10}}{10^2} - \frac{x^{14}}{14^2} + \&c. = (\Sigma)\int x^{-1}\dot{x}\sqrt{\int\frac{\dot{x}}{1+x^4}}.$$

&c.

And thus may we inveſtigate as many ſeries of this claſs as we pleaſe with fluxionary expreſſions for their ſums; the fluents of which may generally be exhibited by the areas of tranſcendant hyperbolas, and in particular caſes by circular arcs. But as this is the moſt troubleſome part of the buſineſs, we ſhall juſt point out an eaſy proceſs in the finding of the fluent of $x^{-1}\dot{x}\int\frac{\dot{x}}{1+x}$ by means of the circular arc, when the ultimate value of x is ſuppoſed to be 1; by which procedure may the ſums of various ſeries of this kind be diſcovered, and conſequently the fluents of their correſponding expreſſions, when conſidered under ſuch limitations as ariſe from or appertain to the circle.

The fluent of $x^{-1}\dot{x}\int\frac{\dot{x}}{1+x}$ is expreſſed by the area of the tranſcendant hyberbola GMLC (ſeries p. 118.) the method of computing which (when x is $= 1$) is there exemplified. But the fluent of this expreſſion may be more commodiouſly exhibited for computation when x has the above value by the arc of a circle. For if the coſine of the circular arc (a) be denoted by y, $\sqrt{-1}\,.\,a$ will be equal $L\,.\,y+\sqrt{y^2-1}$ (ſee Simpſon's Miſcell. Tracts, p. 77.), and by N° 291. (ſeries) $L\,.\,x$ is $= \frac{x^2-1}{x} - \frac{x^4-1}{2x^2} + \frac{x^6-1}{3x^3} - \&c.$ writing therefore $y+\sqrt{y^2-1}$ for x in this ſeries, and reducing the terms, &c. we have

$$\tfrac{1}{2}\sqrt{-1}\,.\,a = \sqrt{y^2-1} - \frac{2y\sqrt{y^2-1}}{2} + \frac{3y^2\sqrt{y^2-1}-y^2-1\big]^{\frac{3}{2}}}{3} - \frac{4y^3\sqrt{y^2-1}-4y.\overline{y^2-1}\big]^{\frac{5}{2}}}{4} + \&c.$$

Multiply the ſides of this equation by $\frac{\dot{a}}{\sqrt{-1}}$ and $\frac{-\dot{y}}{\sqrt{y^2-1}}$ reſpectively, (the latter quantity being the fluxion of the circular arc, rad. 1, coſine y,) and taking the correct fluent, a being evidently $= 0$ when $y = 1$, we have

$$\tfrac{1}{4}a^2 = -\overline{y-1} + \frac{2y^2-2}{2^2} - \frac{y^3-3y.\overline{1-y^2}-1}{3^2} + \frac{y^4-6y^2.\overline{1-y^2}+\overline{1-y^2}\big]^2-1}{4^2} - \frac{y^5-10y^3.\overline{1-y^2}+5y.\overline{1-y^2}\big]^2-1}{5^2} + \&c.$$

Now take y (the cosine) $= 0$, and consequently $a =$ the quadrantal arc (A), and the result is $1 - \frac{2}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{2}{6^2} + \frac{1}{7^2} + \frac{1}{9^2} -$ &c. $=$

$1 - \frac{1}{2^2} - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} - \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{1}{8^2}$ &c. $= \frac{1}{4}A^2$;

which multiplied by $\frac{4}{3}$, and the terms brought to order, becomes

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} -$ &c. $= \frac{1}{3}A^2$, which is therefore equal to

$\int x^{-1}\dot{x} \int \frac{\dot{x}}{1+x}$ in the circumstances above-mentioned.

It is observable that the above literal series $y - \frac{\overline{y-1}}{2^2} + \frac{2y^2-2}{2^2} -$ &c. is equal the difference of the series $y - \frac{y^2-1+y^2}{2^2} + \frac{y^3-3y\cdot\overline{1-y^2}}{3^2} -$ &c. and $1 - \frac{1}{2^2} + \frac{1}{3^2} -$ &c.

that is, $y - \frac{y^2-1+y^2}{2^2} + \frac{y^3-3y\cdot\overline{1-y^2}}{3^2} -$ &c. $- \frac{1}{3}A^2 = -\frac{1}{4}a^2$. And (from Mr. Emerson's Trigon. prop. 26. p. 50.) it appears, that the numerators of the terms of this series are the respective cosines of a, $2a$, $3a$, $4a$, &c. (rad. 1); from which circumstance Mr. Landen in his summations of these kind of series (Memoirs, p. 68.) has introduced the sines and cosines of multiple arcs; from the well-known values of which, when y (the cosine) is 0, $+1$, -1, &c. it more readily appears in what cases the numerators of the terms of the resulting series will become unity, having the signs positive or alternately $+$ and $-$; but it is very evident that this consideration is not absolutely necessary in the summation, the same conclusions being attainable equally without it. The principal and perhaps only advantage arising therefrom is, that we more easily discover by this means than any other the sum of the series when the sign of every other term is changed; the cosines of the arcs a, $2a$, $3a$, &c. being well known to be the cosines of the supplemental arcs $\overline{2A-a}$, $2 \times \overline{2A-a}$, $3 \times \overline{2A-a}$, &c. having their signs $-$ and $+$ alternately, so long as the former arcs do not exceed a quadrant.

Now, from the above series $y - \frac{y^2-1+y^2}{2^2} +$ &c. $= \frac{1}{3}A^2 - \frac{1}{4}a^2$, we have, by multiplying the sides of the equation by $\frac{-\dot{y}}{\sqrt{1-y^2}}$ and $\dot{a}$ respectively, and taking the fluents,

$$\sqrt{1-y^2} - \frac{2y\cdot\sqrt{1-y^2}}{2^3} + \frac{3y^2\cdot\sqrt{1-y^2}-\overline{1-y^2}^{\frac{3}{2}}}{3^3} - \text{&c.} = \frac{1}{3}A^2\,a - \frac{a^3}{12}.$$

And

And if we take y (the cosine) $= 0$, a will become A; and therefore

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \&c. = \tfrac{1}{4}A^3. \quad \text{(See Series, p. 169)}$$

Multiplying again the sides of the last equation but one by $-\frac{\dot{y}}{\sqrt{1-y^2}}$ and $\dot{a}$ as before, and taking their fluents, there arises

$$-y + \frac{y^2 - 1 + y^2}{2^4} - \frac{y^3 - 3y \cdot \overline{1-y^2}}{3^4} + \&c. = -\frac{a^4}{48} + \frac{A^2a^2}{6}.$$

But when y (the cosine) is $= 1$, a will be 0; hence the correct fluent is

$$-\overline{y-1} + \frac{2y^2-2}{2^4} - \frac{y^3 - 3y \cdot \overline{1-y^2+1}}{3^4} - \&c. = \frac{A^2a^2}{6} - \frac{a^4}{48}.$$

Now take $y = 0$, and consequently $a = A$, and the result is

$1 - \frac{2}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} - \frac{2}{6^4} + \&c. = \frac{7A^4}{48}$; which multiplied by $\frac{16}{15}$, and the terms brought to order, as before, becomes

$$1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \&c. = \frac{7A^4}{45}.$$

&c.

By a similar procedure with different fluents, as $\int \frac{\dot{x}}{1+x^2}$, $\int \frac{\dot{x}}{1+x^4}$, &c. may other series be summed, without the consideration of multiple arcs, or any peculiar method of determining the value of the correction of the last fluent; but perhaps not in so perspicuous a manner. And from what we have before observed, that if y be the cosine of the least arc a, $-y$ will be the cosine of the arc $\overline{2A-a}$, we may, by substituting these values for y and a in the above series, derive a variety of others with their summations.

This method (with the assistance of multiple arcs) is farther applied in the beforementioned treatise to series whose denominators consist of the same powers of two or more factors in the same arith. progression; but it becomes more perplexed and embarrassed, and more liable to exceptions. One principal objection is, that when the series is brought to the proposed form by continually multiplying by the same or some other fluxionary expression, and taking the correct fluents, the last correction of the infinite (or series) part of the fluent must be had in known terms before the general expression can be exhibited; unless that correction be exterminated in "bringing the two series together in order to form only one series," but in this case you do not obtain from the general expression the sum of that series into which you had ultimately resolved the equation by taking the fluent. Investigations of the general

theorems, Art. 11, 13, (Memoirs, p. 74, &c.) will ſufficiently explain this obſervation.

Let the two ſeries which expreſs the value of L . $\overline{1+x}$ (N^{os} 292, 319,) be equated, and ſo adjuſted as to vaniſh when x is $= 1$; multiply the equation by $\dot{x}$, and take the correct fluent; multiply again the reſult by $\frac{\dot{x}}{x}$ and correct the fluent; ſubtract one ſide of the equation from the other, and multiply the remainder by $\frac{1}{2\sqrt{-x}}$ and there ariſes by tranſpoſition

$$\frac{1}{2\sqrt{-1}} \times : \frac{x^{\frac{3}{2}} - x^{-\frac{3}{2}}}{1 \cdot 2^2} - \frac{x^{\frac{5}{2}} - x^{-\frac{5}{2}}}{2^2 \cdot 3^2} + \frac{x^{\frac{7}{2}} - x^{-\frac{7}{2}}}{3^2 \cdot 4^2} - \&c. =$$

$$\frac{1}{2\sqrt{-1}} \times : \int \frac{\dot{x}}{x} \int \dot{x} \int \frac{\dot{x}}{x} \int \frac{\dot{x}}{x} + 2\ddot{p}x - 2\dot{p} + \dot{R} + \ddot{R} \cdot X - 2\ddot{p}X - \tfrac{1}{2}X^2;$$ where $\ddot{p}$, $\dot{R}$, $\ddot{R}$ are the reſpective corrections of the ſeries part of the ſucceſſive fluents, the fourth correction S being exterminated in the ſubtraction of the expreſſions; and

$$X = \int \frac{\dot{x}}{x} = L \cdot x.$$

Now for $\int \frac{\dot{x}}{x} \int \dot{x} \int \frac{\dot{x}}{x} \int \frac{\dot{x}}{x}$ write its equal $\frac{X^2 x}{2} - 2Xx - X + 3x - 3$* (by Form L.); alſo for X write $\sqrt{-1} \cdot a$; for $\ddot{p}$, $\frac{A^2}{3}$; and for $\ddot{R} + \dot{R}$, $\frac{2A^2}{3} - 1$, and the latter ſide of the equation reduces to

$$\frac{x^{-\frac{1}{2}}}{2\sqrt{-1}} \times 3x - 3 + \frac{2A^2 x}{3} - \frac{2A^2}{3} - \frac{a^2 x}{2} + \frac{a^2}{2} - \frac{x^{-\frac{1}{2}}}{2\sqrt{-1}} \times 2\sqrt{-1} \cdot ax + 2\sqrt{-1} \cdot x$$

$$= \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{2\sqrt{-1}} \times \frac{2A^2}{3} - \frac{a^2}{2} + 3 - \overline{x^{\frac{1}{2}} + x^{-\frac{1}{2}}} \times a.$$ Laſtly, for x write $y + \sqrt{y^2 - 1}$, and we have $\frac{\overline{y+\sqrt{y^2-1}}^{\frac{3}{2}} - \overline{y-\sqrt{y^2-1}}^{\frac{3}{2}}}{2\sqrt{-1} \cdot 1 \cdot 2^2} - \frac{\overline{y+\sqrt{y^2-1}}^{\frac{5}{2}} - \overline{y-\sqrt{y^2-1}}^{-\frac{5}{2}}}{2\sqrt{-1} \cdot 2^2 \cdot 3^2} + \&c.$

$$= \frac{\overline{y+\sqrt{y^2-1}}^{\frac{1}{2}} - \overline{y-\sqrt{y^2-1}}^{-\frac{1}{2}}}{2\sqrt{-1}} \times \frac{2A^2}{3} - \frac{a^2}{2} + 3 - \overline{y+\sqrt{y^2-1}}^{\frac{1}{2}} + \overline{y-\sqrt{y^2-1}}^{-\frac{1}{2}} \times a,$$

* The theorem in Mr. Landen's Memoirs, p. 138. which ſhould exhibit the fluents of theſe compound fluxional expreſſions, is erroneous in every term after the ſecond.

This theorem, however, may be found truly expreſſed in Simpſon's Fluxions, p. 390. which is eſſentially the ſame with Form L. in the following table.

which

which is evidently Mr. Landen's general theorem, p. 75.; an expreſſion that does not ſeem reducible by any means to the numerators being unity and the ſigns alternately + and −, which is the form of the ſeries part of the laſt fluent when the ultimate value of x is 1, and to which it is well known the expreſſion $\int \frac{\dot{x}}{x} \int \dot{x} \int \frac{\dot{x}}{x} \int \frac{\dot{x}}{1+x}$ is equal, ſuppoſing the fluents to be generated while x from 0 becomes 1. The reaſon of this is indeed evident; for ſince the laſt correction (S) is exterminated by taking one expreſſion from the other, the ſeries is therefore entirely deſtroyed thereby in the equation, and conſequently cannot be deduced therefrom by taking the variable parts of the equation = 0. We may however derive the ſeries when the ſigns are all affirmative, by ſubſtituting $2A-a$ for a in the above equation, which will then be transformed to

$$\frac{\overline{y+\sqrt{y^2-1}}^{\frac{3}{2}} + \overline{y-\sqrt{y^2-1}}^{-\frac{3}{2}}}{2 \cdot 1 \cdot 2^2} - \frac{\overline{y+\sqrt{y^2-1}}^{\frac{5}{2}} + \overline{y-\sqrt{y^2-1}}^{-\frac{5}{2}}}{2 \cdot 2^2 \cdot 3^2} + \&c.$$

$$= \frac{\overline{y+\sqrt{y^2-1}}^{\frac{3}{2}} - \overline{y-\sqrt{y^2-1}}^{-\frac{3}{2}}}{2\sqrt{-1}} \times \overline{4A-2a + \overline{y+\sqrt{y^2-1}}^{\frac{1}{2}} + \overline{y-\sqrt{y^2-1}}^{-\frac{1}{2}}} \times$$

$$\overline{\frac{2A^2}{3} - Aa + \frac{a^2}{4} - \frac{3}{2}};$$

in which taking y (the coſine) = 1, and conſeq. $a = 0$, we have

$$\frac{1}{1 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{3^2 \cdot 4^2} + \&c. = \frac{4A^2}{3} - 3.$$

In Art. 13. (Memoirs) by proceeding from the two values of $\int \frac{\dot{x}}{1+x}$ (Nos 320, 292.) as the latter ſide of the equation denotes, we have

$$\frac{x^3}{1 \cdot 3^2} - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^5}{3^2 \cdot 5^2} - \&c. - \frac{\ddot{p}x^2}{4} + \frac{\ddot{p}}{2}\int \frac{\dot{x}}{x} - \dddot{R}\int \frac{\dot{x}}{x} + \frac{\ddot{p}}{4} - \ddot{S} =$$

$$- \frac{x^{-1}}{1 \cdot 3^2} + \frac{x^{-2}}{2^2 \cdot 4^2} - \&c. + \left.\begin{array}{l} \int \frac{\dot{x}}{x}\int x\dot{x}\int \frac{\dot{x}}{x}\int \frac{\dot{x}}{x} + \frac{1}{4}\int \frac{\dot{x}}{x}\int \frac{\dot{x}}{x} - x + 1 + \frac{\ddot{p}x^2}{4} \\ - \frac{\ddot{p}}{4} - \overset{IV}{R}\int \frac{\dot{x}}{x} - \frac{1}{2}\ddot{p}\int \frac{\dot{x}}{x} + \int \frac{\dot{x}}{x} + \ddot{S} \end{array}\right\} = \left\{\begin{array}{l} \\ \\ \end{array}\right. \overline{\int \frac{\dot{x}}{x}\int x\dot{x}\int \frac{\dot{x}}{x}\int \frac{\dot{x}}{1+x}}.$$

From whence, by ſubſtituting the values of theſe fluentials (Form L.), ſubtracting, dividing by x, and tranſpoſing, we have

$$\frac{x^2+x^{-2}}{1 \cdot 2^2} - \frac{x^3+x^{-3}}{2^2 \cdot 4^2} + \frac{x^4+x^{-4}}{3^2 \cdot 5^2} - \&c. =$$

$$\overline{\frac{x}{8} + \frac{1}{8x} \cdot X^2 - \frac{\ddot{p}}{x} - \frac{\dddot{R}}{x} + \frac{\overset{IV}{R}}{x} - \frac{7}{8x} + \frac{x}{4} \cdot X + \frac{\ddot{p}x}{2} - \frac{\ddot{p}}{2x} + \frac{3x}{16} + \frac{13}{16x} - 1 + 2\ddot{S}x};$$

where

where $\overset{''}{p}$, $\overset{'''}{R}$, $\overset{IV}{R}$, and $\overset{''}{S}$, denote the reſpective corrections of the ſucceeding fluents, and X as before. Now from hence let us endeavour to deduce Mr. L's general theorem (p. 77.) by ſubſtituting according to his method, $\sqrt{-1}\,.\,a$ for X; $\frac{A^2}{3}$ for $\overset{''}{p}$; and $2\overset{''}{S}$ for $\overset{'''}{R} - \overset{IV}{R}$, and we get

$$\frac{A^2x}{6} - \frac{A^2}{6x} + \frac{13}{16x} + \frac{3x}{16} - \frac{1}{2} + \frac{A^2a}{3x\sqrt{-1}} - \frac{7a}{8x\sqrt{-1}} - \frac{2\overset{''}{S}a}{x\sqrt{-1}} + \frac{xa}{4\sqrt{-1}} - \frac{a^2x}{8} - \frac{a^2}{8x} + \frac{2\overset{''}{S}}{x}.$$

Here then it is evident that $\overset{''}{S}$ is not exterminated, therefore its value muſt neceſſarily be known before we can deduce the theorem adverted to; accordingly writing $\frac{A^2}{6} - \frac{5}{16}$* for $\overset{''}{S}$, we have

$$\frac{A^2a}{3x\sqrt{-1}} - \frac{7a}{8x\sqrt{-1}} - \frac{2\overset{''}{S}a}{x\sqrt{-1}} = -\frac{a}{4\sqrt{-1}\,.\,x}, \text{ and } \frac{2\overset{''}{S}}{x} - \frac{A^2}{6x} + \frac{13}{16x} = +\frac{A^2}{6x} + \frac{3}{16x},$$

and therefore by ſubſtituting we get

$$\frac{x^3+x^{-2}}{1\cdot 2^2} - \frac{x^3+x^{-3}}{2^2\cdot 4^2} + \&c. = \frac{x-x^{-1}}{\sqrt{-1}}\cdot\frac{a}{4} + \sqrt{x+x^{-1}} \times \sqrt{\frac{A^2}{6} - \frac{a^2}{8} + \frac{3}{16} - \frac{1}{2}}\,;$$ or,

by writing $y + \sqrt{y^2-1}$ for x, and dividing the whole by 2,

$$\frac{y^2-1+y^2}{1\cdot 2^2} - \frac{y^3-3y\cdot\sqrt{1-y^2}}{2^2\cdot 4^2} + \frac{y^4-6y^2\cdot\sqrt{1-y^2}+\sqrt{1-y^2}^2}{3^2\cdot 5^2} - \&c. =$$

$$\sqrt{1-y^2} \times \frac{a}{4} + y \times \sqrt{\frac{A^2}{6} - \frac{a^2}{8} + \frac{3}{16} - \frac{1}{2}}\,;$$ which is obviouſly Mr. L's theorem, p. 77.

It is therefore very evident that Mr. Landen in the inveſtigations of theſe fundamental theorems has been beholden to ſome proceſs that he has not premiſed (at leaſt in theſe Memoirs), the ſum of the ſeries $\frac{1}{1^2\cdot 3^2} - \frac{1}{2^2\cdot 4^2} + \&c.$ being requiſite to be known previous to obtaining the theorem, p. 77. and that of $\frac{1}{1^2\cdot 3^2\cdot 5^2} -$

* By N° 130. we have $\frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \frac{1}{5^2\cdot 7^2} + \&c. = \frac{A^2}{4} - \frac{1}{2}$, and $\frac{1}{2^2\cdot 4^2} + \frac{1}{4^2\cdot 6^2} + \&c. = \frac{1}{16} \times : \frac{1}{1\cdot 2^2} + \frac{1}{2^2\cdot 3^2} + \&c. = \frac{A^2}{12} - \frac{3}{16}$; and therefore by addition and ſubtraction, we have $\frac{1}{1^2\cdot 3^2} + \frac{1}{2^2\cdot 4^2} + \frac{1}{3^2\cdot 5^2} + \&c. = \frac{A^2}{3} - \frac{11}{16}$, and $\frac{1}{1\cdot 3^2} - \frac{1}{2^2\cdot 4^2} + \frac{1}{3^2\cdot 5^2} - \&c. = \frac{A^2}{6} - \frac{5}{16}$.

$\frac{1}{2^2\cdot 4^2\cdot 6^2}$

$\frac{1}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1}{3^2 \cdot 5^2 \cdot 7^2}$ — &c. previous to raiſing the theorem, p. 79, and this muſt neceſſarily be always the caſe when the laſt correction (S) is not exterminated; and yet theſe very ſeries are the examples exhibited as the reſults of thoſe general theorems, when x is taken $= 1$, ! Is not this what a Logician would call a *Fallacy*,—*A Begging of the Queſtion?* Perhaps Mr. L. may think proper to ſet this matter in a clearer light in ſome future publication; as it rather appears, that the ſums of the above particular ſeries ſhould be always the neceſſary *poſtulata* for obtaining thoſe general theorems; and conſequently that the method of ſumming them ought to have been previouſly conſidered.

By means of the Forms LIX. LX. &c. in the following table, will the ſummations of a different order of ſeries ariſe, *viz.*

327. (F) $\frac{1}{2}x - \frac{3}{2\cdot4} + \frac{1}{3\cdot4}\cdot x^3 + \frac{3\cdot5}{2\cdot4\cdot6} + \frac{3}{3\cdot2\cdot6} + \frac{1}{5\cdot6}\cdot x^5 - \frac{3\cdot5\cdot7}{2\cdot4\cdot6\cdot8} + \frac{3\cdot5}{3\,2\cdot4\cdot8} + \frac{3}{5\cdot2\cdot8} + \frac{1}{7\cdot8}\cdot x^7 + \frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10} + \frac{3\cdot5\cdot7}{3\cdot2\cdot4\,6\cdot10} + \frac{3\cdot5}{5\cdot2\cdot4\cdot10} + \frac{3}{7\cdot2\cdot10} + \frac{1}{9\cdot10}\cdot x^9$ — &c.

(Σ) $\frac{1}{\sqrt{1+x^2}}a + \frac{1}{x}\sqrt{1 - \frac{x^2}{1+x^2}} - \frac{1}{x}$.

328. (F) $\frac{1}{2}x - \frac{3}{2\cdot6} + \frac{1}{3\cdot6}\cdot x^5 + \frac{3\cdot5}{2\cdot4\cdot10} + \frac{3}{3\cdot2\cdot10} + \frac{1}{5\cdot10}\cdot x^9 - \frac{3\cdot5\cdot7}{2\cdot4\cdot6\cdot14} + \frac{3\cdot5}{3\cdot2\cdot4\cdot14} + \frac{3}{5\cdot2\cdot14} + \frac{1}{7\cdot14}\cdot x^{13} + \frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot18} + \frac{3\,5\cdot7}{3\cdot2\cdot4\cdot6\cdot18} + \frac{3\cdot5}{5\cdot2\cdot4\cdot18} + \frac{3}{7\cdot2\cdot18} + \frac{1}{9\cdot18}\cdot x^{17}$ — &c.

(Σ) $\frac{x^{-1}}{\sqrt{1+x^4}}a + \frac{1}{x^3}\sqrt{1 - \frac{x^4}{1+x^4}} - \frac{1}{x^3}$.

329. (F) $\frac{1}{2}x - \frac{3}{2\cdot8} + \frac{1}{3\cdot8}\cdot x^7 + \frac{3\cdot5}{2\cdot4\cdot14} + \frac{3}{3\cdot2\cdot14} + \frac{1}{5\cdot14}\cdot x^{13} - \frac{3\cdot5\cdot7}{2\cdot4\cdot6\cdot20} + \frac{3\cdot5}{3\cdot2\cdot4\cdot20} + \frac{3}{5\cdot2\cdot20} + \frac{1}{7\cdot20}\cdot x^{19} + \frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot26} + \frac{3\cdot5\cdot7}{3\cdot2\cdot4\cdot6\cdot26} + \frac{3\cdot5}{5\cdot2\cdot4\cdot26} + \frac{3}{7\cdot2\cdot26} + \frac{1}{9\cdot26}\cdot x^{25}$ — &c.

(Σ) $\frac{x^{-2}}{\sqrt{1+x^6}}a + \frac{1}{x^5}\sqrt{1 - \frac{x^6}{1+x^6}} - \frac{1}{x^5}$.

&c.

Where the law of continuation is manifeſt; a being the circ. arc, rad. 1. tang. reſpectively, x, x^2, x^3, &c.

From hence may a number of very difficult ſeries be ſummed by taking different values of a. For inſtance, let $a = \frac{1}{3}$A (A being the quadrantal arc, rad. 1.) and x will be $\frac{1}{\sqrt{3}}$; which values being ſubſtituted in N° 327. there ariſes

330. (F) $\frac{1}{2\cdot3^{\frac{1}{2}}} - \frac{3}{2\cdot4} + \frac{1}{3\cdot4} \times \frac{1}{3^{\frac{3}{2}}} + \frac{3\cdot5}{2\cdot4\cdot6} + \frac{3}{3\cdot2\cdot6} + \frac{1}{5\cdot6} \times \frac{1}{3^{\frac{5}{2}}}$ — &c. $= (\Sigma)\ \frac{\sqrt{3}\cdot A}{6} - \sqrt{3} + \frac{3}{2}$.

Again,

Again, let $a = \frac{2}{3}A$, and x will be $3^{\frac{1}{2}}$; hence N° 328. becomes

331. (F) $\frac{1}{2}\times 3^{\frac{1}{2}} - \frac{3}{2\cdot 6} + \frac{1}{3\cdot 6}\times 3^{\frac{5}{2}} + \frac{3\cdot 5}{2\cdot 4\cdot 10} + \frac{3}{3\cdot 2\cdot 10} + \frac{1}{5\cdot 10}\times 3^{\frac{9}{2}} - \&c.$

(Σ) $\frac{1}{3^{\frac{1}{2}}}A - \frac{1}{2\cdot 3^{\frac{3}{2}}}$.

&c.

If a be $= \frac{1}{2}A$, x will be 1, and the sum of each series $= \frac{A}{2\sqrt{2}} + \frac{1-\sqrt{2}}{\sqrt{2}}$, from whence we have this remarkable conclusion, that the infinite series

$\frac{1}{2} - \frac{3}{2\cdot 4} + \frac{1}{3\cdot 4} + \frac{3\cdot 5}{2\cdot 4\cdot 6} + \frac{3}{3\cdot 2\cdot 6} + \frac{1}{5\cdot 6} - \&c.$ is $= \frac{1}{2} - \frac{3}{2\cdot 6} + \frac{1}{3\cdot 6} +$

$\frac{3\cdot 5}{2\cdot 4\cdot 10} + \frac{3}{3\cdot 2\cdot 10} + \frac{1}{5\cdot 10} - \&c. = \frac{1}{2} - \frac{3}{2\cdot 8} + \frac{1}{3\cdot 8} + \frac{3\cdot 5}{2\cdot 4\cdot 14} + \frac{3}{3\cdot 2\cdot 14} + \frac{1}{5\cdot 14} - \&c.$

ad infinitum.

The above theorems are thus inveſtigated. By Form IX. (in the following table) we have $\int \frac{\dot{x}}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c.$ = (*a*) circ. arc, r. 1, t. x, which multiplied by $\frac{\dot{x}}{\overline{1+x^2}^{\frac{3}{2}}}$ (the fluxion of the ſine $\frac{x}{\sqrt{1+x^2}}$) becomes

$\frac{x\dot{x}}{\overline{1+x^2}^{\frac{3}{2}}} - \frac{x^3\dot{x}}{3\cdot\overline{1+x^2}^{\frac{3}{2}}} + \frac{x^5\dot{x}}{5\cdot\overline{1+x^2}^{\frac{3}{2}}} - \&c. = \frac{a\dot{x}}{\overline{1+x^2}^{\frac{3}{2}}}$. Now, the fluents of theſe terms are reſpectively

$$\int \frac{x\dot{x}}{\overline{1+x^2}^{\frac{3}{2}}} = \frac{x^2}{2} - \frac{3x^4}{2\cdot 4} + \frac{3\cdot 5x^6}{2\cdot 4\cdot 6} - \&c.$$

$$-\frac{1}{3}\times\int \frac{x^3\dot{x}}{\overline{1+x^2}^{\frac{3}{2}}} = \frac{1}{3}\times : -\frac{x^4}{4} + \frac{3x^6}{2\cdot 6} - \&c.$$

$$+\frac{1}{5}\times\int \frac{x^5\dot{x}}{\overline{1+x^2}^{\frac{3}{2}}} = \frac{1}{5}\times : \quad + \frac{x^6}{6} - \&c.$$

&c.

Therefore $\frac{x^2}{2} - \frac{3}{2\cdot 4} + \frac{1}{3\cdot 4}\times x^4 + \&c. = \int \frac{a\dot{x}}{\overline{1+x^2}^{\frac{3}{2}}}$; the correct fluent of which is (by Form. LX.) $\frac{x}{\sqrt{1+x^2}}a + \sqrt{1-\frac{x^2}{1+x^2}} - 1$. In like manner are the other theorems inveſtigated by means of the fluents $\int \frac{x\dot{x}}{1+x^4}$, $\int \frac{x^2\dot{x}}{1+x^6}$, &c.

A TA-

A TABLE OF FLUXIONS, WITH THEIR FLUENTS CALCULATED:

Intended to facilitate the Computations of the Sums of the Series from N° 262. to 269. inclusive, when they are attainable only by Means of Circular Arcs and Logarithms; the Formulæ, in such Case, being, in general, ultimately resolvable into these Fluxional Expressions, and Algebraic Quantities.

L, denotes the hyperbolic logarithm; A, the circular arc, rad. 1; s, the sine; t, the tangent; $a = \frac{1}{4}\sqrt{10+2\sqrt{5}}$, $b = \frac{1}{4}\sqrt{10-2\sqrt{5}}$, $c = \frac{1}{4}\times\sqrt{5-1}$. $d = \frac{1}{4}\times\sqrt{5+1}$.

I. $\int \frac{x^n \dot{x}}{1-x} = -\mathrm{L}.\overline{1-x} - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c. - \frac{x^n}{n}$.

II. $\int \frac{x^n \dot{x}}{1+x} = \pm\mathrm{L}.\overline{1+x} \mp x \pm \frac{x^2}{2} \mp \frac{x^3}{3} \pm \frac{x^4}{4} \mp \&c. + \frac{x^n}{n}$.

Where the higher or lower sign obtains according as n is even or odd.

III. $\int \frac{x^{n-1}\dot{x}}{1\mp x^n} = \mp\frac{1}{n}\mathrm{L}.\overline{1\mp x^n}$.

IV. $\int \frac{\dot{x}}{1-x^2} = \frac{1}{2}\mathrm{L}.\frac{1+x}{1-x}$.

V. $\int \frac{\dot{x}}{1-x^3} = \frac{1}{3}\times : \frac{1}{2}\mathrm{L}.\overline{1+x+x^2} - \mathrm{L}.\overline{1-x} + \sqrt{3}.\mathrm{A},\ s.\ \frac{\sqrt{3}.x}{2\sqrt{1+x+x^2}}$.

VI. $\int \frac{\dot{x}}{1-x^4} = \frac{1}{4}\times : \mathrm{L}.\frac{1+x}{1-x} + 2\mathrm{A},\ t.\ x$.

VII. $\int \frac{\dot{x}}{1-x^5} = \begin{cases} \frac{1}{5}\times : -\mathrm{L}.\overline{1-x} - c\,\mathrm{L}.\overline{1-2cx+x^2} - d\,\mathrm{L}.\overline{1+2dx+x^2} \\ +2a\,\mathrm{A},\ s.\ \frac{ax}{\sqrt{1-2cx+x^2}} + 2b\,\mathrm{A},\ s.\ \frac{bx}{\sqrt{1+2dx+x^2}}. \end{cases}$

VIII. $\int \frac{\dot{x}}{1-x^6} = \frac{1}{6}\times : \frac{1}{2}\mathrm{L}.\frac{1+x+x^2}{1-x+x^2} + \mathrm{L}.\frac{1+x}{1-x} + \sqrt{3}.\mathrm{A},\ t.\ \frac{\sqrt{3}.x}{1-x^2}$.

X. $\int \frac{\dot{x}}{1+x^2} = \mathrm{A,}\ t\ .\ x.$

X. $\int \frac{\dot{x}}{1+x^3} = \frac{1}{3} \times : \mathrm{L}\ .\ \overline{1+x} - \frac{1}{2}\mathrm{L}\ .\ \overline{1-x+x^2} + \sqrt{3}\ .\ \mathrm{A,}\ s\ .\ \frac{\sqrt{3}\ .\ x}{2\sqrt{1-x+x^2}}.$

XI. $\int \frac{\dot{x}}{1+x^4} = \frac{1}{4} \times : \sqrt{\frac{1}{2}}\ \mathrm{L}\ .\ \frac{1+\sqrt{2}\ .\ x+x^2}{1-\sqrt{2}\ .\ x+x^2} + \sqrt{2}\ .\ \mathrm{A,}\ t\ .\ \frac{\sqrt{2}\ .\ x}{1-x^2}.$

XII. $\int \frac{\dot{x}}{1+x^5} = \begin{cases} \frac{1}{5} \times : \mathrm{L}\ .\ \overline{1+x} - d\,\mathrm{L}\ .\ \overline{1-2dx+x^2} - c\,\mathrm{L}\ .\ \overline{1+2cx+x^2} \\ +2b\ \mathrm{A,}\ s\ .\ \frac{bx}{\sqrt{1-2dx+x^2}} + 2a\ \mathrm{A,}\ s\ .\ \frac{ax}{\sqrt{1+2cx+x^2}}. \end{cases}$

XIII. $\int \frac{\dot{x}}{1+x^6} = \begin{cases} \frac{1}{6} \times : -\sqrt{\frac{3}{4}}\ .\ \mathrm{L}\ .\ \overline{1-\sqrt{3}\ .\ x+x^2} - \sqrt{\frac{3}{4}}\ .\ \mathrm{L}\ .\ \overline{1+\sqrt{3}\ .\ x+x^2} \\ +\mathrm{A,}\ s\ .\ \frac{x}{2\sqrt{1-\sqrt{3}\ .\ x+x^2}} + 2\mathrm{A,}\ t\ .\ x + \mathrm{A,}\ s\ .\ \frac{x}{2\sqrt{1+\sqrt{3}\ .\ x+x^2}}. \end{cases}$

XIV. $\int \frac{x\dot{x}}{1-x^3} = \frac{1}{3} \times : \frac{1}{2}\mathrm{L}\ .\ \overline{1+x+x^2} - \mathrm{L}\ .\ \overline{1-x} - \sqrt{3}\ .\ \mathrm{A,}\ s\ .\ \frac{\sqrt{3}\ .\ x}{2\sqrt{1+x+x^2}}.$

XV. $\int \frac{x\dot{x}}{1-x^4} = \frac{1}{4} \times : \frac{1}{2}\mathrm{L}\ .\ \frac{1+x^2}{1-x^2}.$

XVI. $\int \frac{x\dot{x}}{1-x^5} = \begin{cases} \frac{1}{5} \times : d\,\mathrm{L}\ .\ \overline{1-2cx+x^2} - c\,\mathrm{L}\ .\ \overline{1+2dx+x^2} - \mathrm{L}\ .\ \overline{1-x} \\ +2b\ \mathrm{A,}\ s\ .\ \frac{ax}{\sqrt{1-2cx+x^2}} - 2a\ \mathrm{A,}\ s\ .\ \frac{bx}{\sqrt{1+2dx+x^2}}. \end{cases}$

XVII. $\int \frac{x\dot{x}}{1-x^6} = \frac{1}{6} \times : \frac{1}{2}\mathrm{L}\ .\ \overline{1+x^2+x^4} - \mathrm{L}\ .\ \overline{1-x^2} + \sqrt{3}\ .\ \mathrm{A,}\ s\ .\ \frac{\sqrt{3}\ .\ x^2}{2\sqrt{1+x^2+x^4}}.$

XVIII. $\int \frac{x\dot{x}}{1+x^3} = \frac{1}{3} \times : \frac{1}{2}\mathrm{L}\ .\ \overline{1-x+x} - \mathrm{L}\ .\ \overline{1+x} + \sqrt{3}\ .\ \mathrm{A,}\ s\ .\ \frac{\sqrt{3}\ .\ x}{2\sqrt{1-x+x^2}}.$

XIX. $\int \frac{x\dot{x}}{1+x^4} = \frac{1}{2}\mathrm{A,}\ t\ .\ x^2.$

XX. $\int \frac{x\dot{x}}{1+x^5} = \begin{cases} \frac{1}{5} \times : c\,\mathrm{L}\ .\ \overline{1+2ax+x^2} - d\,\mathrm{L}\ .\ \overline{1-2cx+x^2} - \mathrm{L}\ .\ \overline{1+x} \\ +2b\ .\ \mathrm{A,}\ s\ .\ \frac{ax}{\sqrt{1-2cx+x^2}} - 2a\ .\ \mathrm{A,}\ s\ .\ \frac{bx}{\sqrt{1+2dx+x^2}}. \end{cases}$

XXI.

XXI. $\int \frac{x\dot{x}}{1+x^6} = \frac{1}{6} \times : \mathrm{L} . \overline{1+x^2} - \frac{1}{2} \mathrm{L} . \overline{1-x^2+x^4} + \sqrt{3} . \mathrm{A}, s . \frac{\sqrt{3} . x^2}{2\sqrt{1-x^2+x^4}}$.

XXII. $\int \frac{x^2\dot{x}}{1-x^4} = \frac{1}{4} \times : \mathrm{L} . \frac{1+x}{1-x} - 2\,\mathrm{A}, t . x$.

XXIII. $\int \frac{x^2\dot{x}}{1-x^6} = \frac{1}{6} \mathrm{L} . \frac{1+x^3}{1-x^3}$.

XXIV. $\int \frac{x^2\dot{x}}{1+x^4} = \begin{cases} \frac{1}{4} \times : \sqrt{\frac{1}{2}} . \mathrm{L} . \frac{1-2^{\frac{1}{2}}x+x^2}{1+2^{\frac{1}{2}}x+x^2} + \sqrt{2} . \mathrm{A}, s . \frac{\sqrt{\frac{1}{2}}}{\sqrt{1-2^{\frac{1}{2}}x+x^2}} \\ +\sqrt{2} . \mathrm{A}, s . \frac{\sqrt{\frac{1}{2}}}{\sqrt{1+2^{\frac{1}{2}}x+x^2}} . \end{cases}$

XXV. $\int \frac{x^2\dot{x}}{1+x^6} = \frac{1}{3} \mathrm{A}, t . x^3$.

XXVI. $\int \frac{x^3\dot{x}}{1-x^6} = \frac{1}{4} \mathrm{L} . \overline{1+x^2}\,\overline{+x^4} - \frac{1}{6} \mathrm{L} . \overline{1-x^6} - \frac{1}{2\sqrt{3}} \mathrm{A}, t . \frac{\sqrt{3} . x^2}{2+x^2}$.

XXVII. $\int \frac{x^3\dot{x}}{1+x^6} = \frac{1}{6} \times : \frac{1}{2} \mathrm{L} . \overline{1-x+x^2} - \mathrm{L} . \overline{1+x} + \sqrt{3} . \mathrm{A}, s . \frac{\sqrt{3}}{\frac{1}{2}\sqrt{1-x+x^2}}$.

XXVIII. $\int \frac{x^4\dot{x}}{1-x^6} = \frac{1}{6} \times : \frac{1}{2} \mathrm{L} . \frac{1+x+x^2}{1-x+x^2} + \mathrm{L} . \frac{1+x}{1-x} - \sqrt{3} . \mathrm{A}, t . \frac{\sqrt{3} . x}{1-x^2}$.

XXIX. $\int \frac{x^4\dot{x}}{1+x^6} = \frac{1}{6} \times : 3\mathrm{A}, t . \frac{x}{1-x^2} - 2\,\mathrm{A}, t . x^3 - \frac{1}{2}\sqrt{3} . \mathrm{L} . \frac{1+\sqrt{3} . x+x^2}{1-\sqrt{3} . x+x^2}$.

The following forms are such as arise in Orders 27, 28, 29, 30, when the series are not reducible into pure algebraic expressions; and as they are here *generally* expressed, some of them will require adjusting to the proposed series, by observing the results of both when x is supposed to vanish.

XXX. $\int \frac{\dot{x}}{\overline{1-x}^{\frac{1}{2}}} = -2\sqrt{1-x}$.

XXXI. $\int \frac{x^{\frac{1}{2}}\dot{x}}{\overline{1-x}^{\frac{1}{2}}} = \mathrm{A}, s . \sqrt{x}, -\sqrt{x-x^2}$.

XXXII. $\int \frac{x\dot{x}}{\overline{1-x}^{\frac{1}{2}}} = -\frac{4}{3}\sqrt{1-x} - \frac{2}{3}\sqrt{x-x^2}.$

XXXIII $\int \frac{x^{\frac{3}{2}}\dot{x}}{\overline{1-x}^{\frac{1}{2}}} = -\frac{1}{2}x^{\frac{3}{2}}\sqrt{1-x} - \frac{3}{4}\sqrt{x-x^2} + \frac{3}{4}A, s.\sqrt{x}.$

XXXIV. $\int \frac{x^{\frac{5}{2}}\dot{x}}{\overline{1-x}^{\frac{1}{2}}} = -\overline{\frac{1}{6}x^{\frac{5}{2}} + \frac{5}{24}x^{\frac{3}{2}} + \frac{5}{16}x^{\frac{1}{2}}} \times \sqrt{1-x} + \frac{5}{16}A, s.\sqrt{x}.$

XXXV. $\int \frac{\dot{x}}{\overline{1-x}^{\frac{3}{2}}} = \frac{2}{\sqrt{1-x}}.$

XXXVI. $\int \frac{x^{\frac{1}{2}}\dot{x}}{\overline{1-x}^{\frac{3}{2}}} = \frac{2x^{\frac{1}{2}}}{\sqrt{1-x}} - 2A, s.\sqrt{x}.$

XXXVII. $\int \frac{x\dot{x}}{\overline{1-x}^{\frac{3}{2}}} = \frac{4-2x}{\sqrt{1-x}}.$

XXXVIII. $\int \frac{x^{\frac{3}{2}}\dot{x}}{\overline{1-x}^{\frac{3}{2}}} = \frac{3x^{\frac{1}{2}}-x^{\frac{3}{2}}}{\sqrt{1-x}} - 3A, s.\sqrt{x}.$

XXXIX. $\int \frac{x^{\frac{5}{2}}\dot{x}}{\overline{1-x}^{\frac{3}{2}}} = \frac{2x^{\frac{7}{2}}}{\sqrt{1-x}} + \overline{2x^{\frac{5}{2}} + \frac{5}{2}x^{\frac{3}{2}} + \frac{15}{4}x^{\frac{1}{2}}} \times \sqrt{1-x} - \frac{15}{4}A, s.\sqrt{x}.$

XL. $\int \frac{\dot{x}}{\overline{1+x}^{\frac{1}{2}}} = 2\sqrt{1+x}.$

XLI. $\int \frac{x^{\frac{1}{2}}\dot{x}}{\overline{1+x}^{\frac{1}{2}}} = \overline{x+x^2}^{\frac{1}{2}} - L.\overline{x^{\frac{1}{2}} + \sqrt{1+x}}.$

XLII. $\int \frac{x\dot{x}}{\overline{1+x}^{\frac{1}{2}}} = \frac{2}{3}\sqrt{x^2+x^3} - \frac{4}{3}\sqrt{1+x}.$

XLIII. $\int \frac{x^{\frac{3}{2}}\dot{x}}{\overline{1+x}^{\frac{1}{2}}} = \frac{3}{4}L.\overline{x^{\frac{1}{2}} + \sqrt{1+x}} + \sqrt{x+x^2} + \frac{1}{2}x^{\frac{3}{2}}.\sqrt{1+x}.$

XLIV. $\int \frac{x^{\frac{5}{2}}\dot{x}}{\overline{1+x}^{\frac{1}{2}}} = -\frac{5}{8}L.\overline{x^{\frac{1}{2}} + \sqrt{1+x}} + \overline{x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{2} + \frac{x^{\frac{5}{2}}}{3}} \times \sqrt{1+x}.$

XLV. $\int \frac{\dot{x}}{\overline{1+x}^{\frac{3}{2}}} = -\frac{2}{\sqrt{1+x}}.$

XLVI.

XLVI. $\int \frac{x^{\frac{1}{2}}\dot{x}}{\overline{1+x}^{\frac{3}{2}}} = 2\,\mathrm{L}.\,\overline{x^{\frac{1}{2}}+\sqrt{1+x}+\sqrt{x+x^2}} + \frac{2x^{\frac{3}{2}}}{\sqrt{1+x}}.$

XLVII. $\int \frac{x\dot{x}}{\overline{1+x}^{\frac{3}{2}}} = \frac{4+2x}{\sqrt{1+x}}.$

XLVIII. $\int \frac{x^{\frac{3}{2}}\dot{x}}{\overline{1+x}^{\frac{3}{2}}} = -3\,\mathrm{L}.\,\overline{x^{\frac{1}{2}}+\sqrt{1+x}+\sqrt{x+x^2}} + \frac{3x^{\frac{3}{2}}}{\sqrt{1+x}}.$

XLIX. $\int \frac{x^{\frac{5}{2}}\dot{x}}{\overline{1+x}^{\frac{3}{2}}} = \frac{15}{4}\,\mathrm{L}.\,\overline{x^{\frac{1}{2}}+\sqrt{1+x}+\sqrt{x+x^2}} + \frac{3x^{\frac{3}{2}}}{\sqrt{1+x}} + \frac{x^{\frac{5}{2}}}{2\sqrt{1+x}}.$

L. $mx^{-m}\int x^{m-1}\dot{x}\overline{\int x^{-1}\dot{x}}^{n} = \left\{\begin{array}{l} \int \overline{x^{-1}\dot{x}}^{n} - \frac{n}{m}\int \overline{x^{-1}\dot{x}}^{n-1} + \frac{n.n-1}{m^2}\int \overline{x^{-1}\dot{x}}^{n-2} \\ -\frac{n.n-1.n-2}{m^3}\int \overline{x^{-1}\dot{x}}^{n-3} + \&c. \end{array}\right.$

LI. $\int \dot{z}\overline{\int y\dot{x}} = z\int y\dot{x} - \int yz\dot{x}.$

LII. $\int \dot{u}\int \dot{z}\int y\dot{x} = \int z\dot{u}.\int yz\dot{x} - \iint z\dot{u}.y\dot{x} + u\int yz\dot{x} - \int uyz\dot{x}.$

Where the relation of u, z, y to x are given.

LIII. $\int \frac{\dot{x}}{x}\int \frac{\dot{x}}{x} = \frac{1}{2}\mathrm{X}^2 - \frac{1}{2}\mathrm{C}^2, = \frac{1}{2}\overline{\mathrm{L}.\frac{x}{c}}^2.$

LIV. $\int \dot{x}\int \frac{\dot{x}}{x} = \overline{\mathrm{X}-1}\times x - \overline{\mathrm{C}-1}\times c.$

LV. $\int \dot{x}\int \frac{\dot{x}}{x}\int \frac{\dot{x}}{x} = \overline{\mathrm{X}^2-\mathrm{C}^2-2\mathrm{X}+2}\times\frac{x}{2} + \overline{\mathrm{C}-1}\times c.$

LVI. $\int \frac{\dot{x}}{x}\int \dot{x}\int \frac{\dot{x}}{x}\int \frac{\dot{x}}{x} = \overline{\mathrm{X}^2-\mathrm{C}^2-4\mathrm{X}+6}\times\frac{x}{2} + \overline{\mathrm{C}-1.\mathrm{X}+3\mathrm{C}-\mathrm{C}^2-3}\times c.$

LVII. $\int x\dot{x}\int \frac{\dot{x}}{x}\int \frac{\dot{x}}{x} = \overline{\mathrm{X}^2-\mathrm{C}^2-\mathrm{X}+\frac{1}{2}}\times\frac{x^2}{4} + \overline{\mathrm{C}-\frac{1}{2}}\times\frac{c^2}{4}.$

LVIII. $\int \frac{\dot{x}}{x}\int x\dot{x}\int \frac{\dot{x}}{x}\int \frac{\dot{x}}{x} = \overline{\mathrm{X}^2-\mathrm{C}^2-2\mathrm{X}+\frac{3}{2}}.\frac{x^2}{8} + \overline{\mathrm{C}-\frac{1}{2}.\mathrm{X}-\mathrm{C}^2+\frac{3}{2}\mathrm{C}-\frac{3}{2}}.\frac{c^2}{4}.$

&c.

Where X is the hyp. log. of x, or fluent of $\frac{\dot{x}}{x}$; c the value of x when the fluent is $= 0$, and C the hyp. log. of c.

If c be taken $= 1$, theſe theorems will be adjuſted to the ſeries in p. 20, &c.

LIX. $\int \overline{a^n \dot{x}} = \begin{cases} xa + n\sqrt{1-x^2}.a^{n-1} - n.n-1.x.a^{n-2} - n.n-1.n-2.\sqrt{1-x^2}.a^{n-3} \\ +n.n-1.n-2.n-3.x.a^{n-4} + n \ldots n-4.\sqrt{1-x^2}.a^{n-5} - \&c. \end{cases}$

a denoting the circular arc, of which the sine is x, and radius 1.

LX. $$\int \frac{a\,i}{\overline{1+t}^{\frac{3}{2}}} = \int \frac{t\,i}{\overline{1+t^2}^{\frac{3}{2}}} - \int \frac{t^3\,i}{3.\overline{1+t}^{\frac{3}{2}}} + \int \frac{t^5\,i}{5.\overline{1+t^2}^{\frac{3}{2}}} - \&c.$$

$$= \frac{t}{\sqrt{1+t^2}}a + \sqrt{1-\frac{t^2}{1+t^2}} - 1.$$

LXI. $$\frac{1}{t^2}\int \frac{a\,t\,i}{\overline{1+t^4}^{\frac{3}{2}}} = \int \frac{t\,i}{\overline{1+t^2}^{\frac{3}{2}}} - \int \frac{t^5\,i}{3.\overline{1+t^4}^{\frac{3}{2}}} + \int \frac{t^9\,i}{5.\overline{1+t^4}^{\frac{3}{2}}} - \&c.$$

$$= \frac{1}{\sqrt{1+t^4}}a + \frac{1}{t^2}\sqrt{1-\frac{t^4}{1+t^4}} - \frac{1}{t^2}.$$

LXII. $$\frac{1}{t^4}\int \frac{a\,t^7\,i}{\overline{1+t^4}^{\frac{3}{2}}} = \int \frac{t\,i}{\overline{1+t^6}^{\frac{3}{2}}} - \int \frac{t^7\,i}{3.\overline{1+t^6}^{\frac{3}{2}}} + \int \frac{t^{13}\,i}{5.\overline{1+t^6}^{\frac{3}{2}}} - \&c.$$

$$= \frac{t^{-1}}{\sqrt{1+t^6}}a + \frac{1}{t^4}\sqrt{1-\frac{t^6}{1+t^6}} - \frac{1}{t^4}.$$

&c.

Where a is the circ. arc, rad. 1, and tang. respectively, t, t^2, t^3, &c.

Generally,

LXIII. $$\frac{1}{t^{n-2}}\int \frac{a\,t^{\frac{n}{2}-1}\,i}{\overline{1+t^n}^{\frac{3}{2}}} = \int \frac{t\,i}{\overline{1+t^n}^{\frac{3}{2}}} - \int \frac{t^{n+1}\,i}{3.\overline{1+t^n}^{\frac{3}{2}}} + \int \frac{t^{2n+1}\,i}{5.\overline{1+t^n}^{\frac{3}{2}}} - \&c.$$

$$= \frac{t^{2-\frac{n}{2}}}{\sqrt{1+t^n}}a + \frac{1}{t^{n-2}}\sqrt{1-\frac{t^n}{1+t^n}} - \frac{1}{t^{n-2}}.$$

a the circ. arc, rad. 1, tang. $t^{\frac{n}{2}}$; and n any even positive number.

EXAM-

EXAMPLES TO SOME OF THE PRECEDING THEOREMS.

1. *Let the ſeries propoſed be*

$$\frac{1}{5.6.7.15} + \frac{1}{6.7.9.17} + \frac{1}{7.8.11.19} + \&c.$$

This ſeries belongs to N° 265. from whence we have by compariſon $p = 4$, $q = n = 1$, $m = r = 5$, $s = u = 2$, $t = 13$, and $\therefore \pi = \frac{5}{2}$, $\delta = \frac{3}{2}$, and $\omega = 4$; which correſponds with the firſt particular caſe, ω and $\overline{\pi - \delta}$ being whole numbers. Hence by ſubſtitution we have

$$(\Sigma)\ \frac{1}{2.4.-\frac{3}{2}.-\frac{5}{2}} \times \overline{\frac{1}{5+2} + \frac{1}{5+4} + \frac{1}{5+6} + \frac{1}{5+8}} + \frac{1}{2.2.\frac{5}{2}.-\frac{3}{2}} \times \frac{1}{5} = \frac{491}{675075};$$

the ſum of the propoſed infinite ſeries.

By comparing the ſame ſeries with N° 272. we find $h = 5$, $e = 1$, $g = 3$; and thence by ſubſtitution derive the ſame concluſion.

2. *Where the ſeries propoſed is* $\frac{1}{9.11.12.28} - \frac{1}{10.12.14.30} + \frac{1}{11.13.16.32} - \&c.$

Here we find $p = 8$, $q = n = 1$, $m = r = 10$, $s = u = 2$, $t = 26$, and $\therefore \pi = 5$, $\delta = 3$, $\omega = 8$, where ω and $\overline{\pi - \delta}$ are even numbers. Hence by the ſame theorem we have

$$(\Sigma)\ \frac{1}{2.8.-3.-5} \times \overline{\frac{1}{12} - \frac{1}{14} + \frac{1}{16} - \frac{1}{18} + \frac{1}{20} - \frac{1}{22} + \frac{1}{24} - \frac{1}{26}} + \frac{1}{2.2.5.-3.2} \times \overline{\frac{1}{9} - \frac{1}{10}}$$

$$= \frac{631}{34594560}.$$

This ſeries being compared with N° 272. we have $h = 10$, $g = 6$, $e = 2$; and the ſum as above.

3. *The ſeries propoſed being* $\frac{1}{1.9 \times 3.5} + \frac{1}{2.11 \times 5.6} + \frac{1}{3.13 \times 7.7} + \&c.$

This ſeries is found to correſpond with the ſecond particular caſe of the above theorem, π, ω and δ being reſpectively = to 4, $\frac{7}{2}$ and $\frac{1}{2}$. From whence by ſubſtitution we have

$$(\Sigma)\ \frac{1}{2.2.4.\frac{1}{2}.\frac{7}{2}} \times \overline{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}} + \frac{1}{2.\frac{7}{2}.\frac{1}{2}.-3} \times \overline{\frac{1}{3} + \frac{1}{5} + \frac{1}{7}} = \frac{7413}{740880}.$$

Comparing the ſame ſeries with N° 273. we have $g = 1$, $e = 3$, $h = 2$; and thence deduce the ſame concluſion.

4. *The*

4. *The ſeries to be ſummed being* $\frac{1}{3\cdot4\cdot6\cdot7} - \frac{1}{4\cdot5\cdot7\cdot8} + \frac{1}{5\cdot6\cdot8\cdot9} - \&c.$

Divide this ſeries by 4, and it becomes $\frac{1}{3\cdot8\cdot12\cdot7} - \frac{1}{4\cdot10\cdot14\cdot8} + \&c.$ which being compared with the ſame formula (265.) we have $p = 2$, $q = 1$, $m = 6$, &c. from whence $\omega = 1$, $\delta = 3$, $\pi = 4$. But here $\overline{\omega - \delta}\ (-2)$ is negative, therefore according to Remark 2. N° 273. write δ for ω, and ω for δ; that is, write m, r, n, s, for r, m, s, n, reſpectively; and by ſubſtitution we ſhall have

$$(\Sigma)\ \frac{1}{2\cdot2\cdot4\cdot3} \times \overline{\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}} + \frac{1}{2\cdot3\cdot-2} \times \overline{\frac{1}{8} - \frac{1}{10}} = \frac{1}{2880};$$

which, being multiplied by 4, produces $\frac{1}{720}$, the ſum of the propoſed ſeries.

In comparing this ſeries with N° 273. we have $(\omega - \delta)\ e$ negative, writing therefore δ for ω, and ω for δ; that is, taking $r = 2h+g-2e+2$, and $m = 2h+g+2$, we get $e = 2$, $g = 2$, and $h = 3$; and the reſult the ſame as above. And thus may the ſum of any other ſeries be found, provided they come under the limitations ſpecified, when the general formulæ ſeem to fail.

5. *Let the ſeries propoſed be* $\frac{3\cdot8x^5}{3\cdot6\cdot12\cdot15} + \frac{5\cdot13x^6}{6\cdot8\cdot15\cdot18} + \frac{7\cdot18x^7}{9\cdot10\cdot18\cdot24} + \&c.$

This ſeries correſponds with N° 269. or N° 270. Comparing the terms with the latter, we have $a = 1$, $b = n = 2$, $c = q = s = u = 3$, $e = 5$, $p = 0$, $m = 4$, $r = 9$, $t = 12$. From whence $\alpha = \frac{1}{2}$, $\beta = \frac{3}{5}$; $\acute{\alpha} = -\frac{3}{2}$, $\acute{\beta} = -\frac{7}{5}$; $\ddot{\alpha} = -\frac{5}{2}$, $\ddot{\beta} = -\frac{12}{5}$; $\dddot{\alpha} = -\frac{7}{2}$, $\dddot{\beta} = -\frac{17}{5}$; $\pi = 2$, $\rho = 3$, $\sigma = 4$; $\acute{\pi} = -2$, $\acute{\rho} = 1$, $\acute{\sigma} = 2$; $\ddot{\pi} = -3$, $\ddot{\rho} = -1$, $\ddot{\sigma} = 1$; $\dddot{\pi} = -4$, $\dddot{\rho} = -2$, $\dddot{\sigma} = -1$; $k = 2$, $l = 3$, and therefore $\frac{w}{v} = \frac{t}{u} = 4$; $\frac{be}{qnsu} = \frac{5}{27}$; hence $A = \frac{1}{432}$, $B = -\frac{7}{72}$, $C = \frac{10}{27}$, $D = -\frac{119}{432}$.

Theſe values being written in the formula (Σ), and X (the ultimate value of x) taken $= 1$, the reſult will be

$$3 \times : \frac{7}{72} + \overline{\frac{1}{3} + \frac{1}{6} - \frac{10}{27}} \times \overline{\frac{1}{3} + \frac{1}{6} + \frac{1}{9}} + \frac{119}{432} \times \overline{\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12}} = \frac{211}{5184},$$

the ſum of the propoſed infinite ſeries when the ultimate value of x is unity.

6. *Re-*

6. *Required the ſum of the ſeries* $\frac{5}{1.2} + \frac{6.3}{2.3.4} + \frac{7.9}{3.4.16} + \frac{8.27}{4.5.64} +$ &c.

Here we find $a = 1$, $b = 1$, $p = 0$, $q = m = n = 1$; and becauſe l is 1, $\frac{w}{v}$ is $= \frac{m}{n} = 1$, and it appears that x is $= \frac{1}{4}$. Therefore in order to adapt this ſeries to the general one, ſuppoſe each term to be multiplied by x^2, and by ſubſtitution we have $\alpha = 4$, $\dot{\alpha} = 3$, $\pi = 1$, $\dot{\pi} = -1$; and $\because A = 4$, $B = -3$; hence $(\Sigma) = \overline{x \times 4 - \frac{3}{x} \times - L.\overline{1-x}} + x^2 \times \frac{3}{x}$, and taking $x = \frac{1}{4}$, and dividing by $\frac{9}{16}(x^2)$ the reſult is 4, the ſum of the propoſed ſeries.

7. *Let the ſeries propoſed be* $\frac{19x^3}{1.2.3} + \frac{28x^4}{2.3.4} + \frac{39x^5}{3.4.5} + \frac{52x^6}{4.5.6} + \frac{67x^7}{5.6.7} + \frac{84x^8}{6.7.8} +$ &c.

This ſeries is evidently reſolvable into theſe two

$$\frac{19x^3}{1.2.3} + \frac{28x^4}{2.3.4} + \frac{37x^5}{3.4.5} + \frac{46x^6}{4.5.6} + \frac{55x^7}{5.6.7} + \text{&c.}$$

$$\frac{1.2x^3}{3.4.5} + \frac{2.3x^4}{4.5.6} + \frac{3.4x^7}{5.6.7} + \frac{4.5x^8}{6.7.8} + \text{&c.}$$

By comparing the former we have $a = 10$, $b = 9$, $p = 0$, $q = m = n = s = 1$, $r = 2$, $l = 2$; and therefore $\frac{w}{v} = \frac{r}{s} = 2$. Hence $\pi = 1$, $\rho = 2$, $A = 5$, $B = -1$, $C = -4$; and by ſubſtitution $(\Sigma) = \overline{x^2 \times 5 - \frac{1}{x} - \frac{4}{x} \times \int \frac{\dot{x}}{1-x}}$

$+ \overline{x^3 \times \frac{1}{x} + \frac{4}{x^2} \times 1 + \frac{x}{2}} = \overline{5x^2 - x - 4} \times - L.\overline{1-x} + 3x^2 + 4x$, the ſum. And in the latter we find $a = 0$, $b = c = e = q = n = s = 1$, $p = 2$, $m = 3$, $r = 4$, $\pi = 1$, $\rho = 2$, $l = 2$; $\because \frac{w}{v} = \frac{r}{s} = 4$, and thence $A = 1$, $B = -6$, $C = 6$; and by ſubſtitution $(\Sigma) = x^4 \times : \overline{\frac{1}{x^2} - \frac{6}{x^3} + \frac{6}{x^4}} \times \int \frac{x^2\dot{x}}{1-x} + x^5 \times \overline{\frac{2}{x} - \frac{6}{x^2} \times \frac{1}{3} + \frac{x}{4}}$

$= $ (becauſe $\int \frac{x^2\dot{x}}{1-x}$ is $= -L.\overline{1-x} - x - \frac{x^2}{2}$, by Form 1.) $\overline{x^2 - 6x + 6} \times -L.\overline{1-x} + 3x^2 - 6x$, the ſum. Hence the ſum of the whole ſeries will be $\overline{2 - 7x + 6x^2} \times -L.\overline{1-x} + 6x^2 - 2x$; which will evidently be algebraical if x be

taken = a root of the equation $6x^2 - 7x + 2 = 0$. Now the two roots of this equation are $\frac{2}{3}$ and $\frac{1}{2}$; the former of which being written for x gives

$$\frac{19 \cdot 2^3}{1 \cdot 2 \cdot 3 \cdot 3^3} + \frac{28 \cdot 2^4}{2 \cdot 3 \cdot 4 \cdot 3^4} + \frac{39 \cdot 2^5}{3 \cdot 4 \cdot 5 \cdot 3^5} + \&c. = 1\tfrac{1}{3},$$

and the latter

$$\frac{19}{1 \cdot 2 \cdot 3 \cdot 2^3} + \frac{28}{2 \cdot 3 \cdot 4 \cdot 2^4} + \frac{39}{3 \cdot 4 \cdot 5 \cdot 2^5} + \&c. = \tfrac{1}{2}.$$

8. *Let the propoſed ſeries be*

$$\text{Circ. arc, s. } \frac{\sqrt{8}-\sqrt{3}}{2 \cdot 3} + \text{arc, s. } \frac{\sqrt{15}-\sqrt{8}}{3 \cdot 4} + \text{arc, s. } \frac{\sqrt{24}-\sqrt{15}}{4 \cdot 5} + \&c.$$

This ſeries is evidently equal to

$$\text{arc, s. } \tfrac{1}{2}\sqrt{\tfrac{8}{9} - \tfrac{1}{3}\sqrt{\tfrac{3}{4}}} + \text{arc, s. } \tfrac{1}{3}\sqrt{\tfrac{5}{16} - \tfrac{1}{4}\sqrt{\tfrac{8}{9}}} + \&c. = \text{arc, s. } \tfrac{1}{2}\sqrt{1-\tfrac{1}{9}-\tfrac{1}{3}\sqrt{1-\tfrac{1}{4}}}$$

$$+ \text{arc, s. } \tfrac{1}{3}\sqrt{1-\tfrac{1}{16}-\tfrac{1}{4}\sqrt{1-\tfrac{1}{9}}} + \&c.$$ Hence, by comparing this with N° 274. we have $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = \frac{1}{4}$, &c. decreaſing quantities; and $r = 1$. Therefore (Σ) = Circ. arc, ſine $\frac{1}{2}$, rad. 1. = ·523598 &c. which agrees with N° 239.

9. *Let the ſeries required to be ſummed be*

$$\text{Circ. arc, t. } \frac{5000}{9950} + \text{arc, t. } \frac{5000}{9851} + \text{arc, t. } \frac{5000}{9753} + \&c.$$

This ſeries appears to be equal to

$$\text{Circ. arc, t. } \frac{10000}{100^2+100\times99} + \text{arc, t. } \frac{10000}{100^2+99\times98} + \text{arc, t. } \frac{10000}{100^2+98\times97} + \&c.$$

which therefore correſponds with N° 276. from whence we find $a = r = 100$; a, b, c, &c. decreaſing quantities; and therefore (Σ) = Circ. arc, tang. 100, rad. 100 = 78·539 &c. is the ſum of the propoſed ſeries.

The theorems to which theſe examples are given, are inveſtigated upon this moſt obvious principle, that if there be a ſeries of quantities a, b, c, d, &c. any how decreaſing till they vaniſh, we ſhall have $\overline{a-b} + \overline{b-c} + \overline{c-d} + \overline{d-e}$ &c. $= a - e =$ (when the laſt quantity e is $= 0$) a. Whence, if a, b, c, &c. denote the ſines of circular arcs, we obtain the firſt theorem; if coſines, the ſecond theorem ariſes, &c.

10. *Required the ſum of the infinite ſeries*

$$1 - \frac{4 \cdot 3^2}{5 \cdot 10 \cdot 5^2} - \frac{4 \cdot 9 \cdot 14 \cdot 3^4}{5 \cdot 10 \cdot 15 \cdot 20 \cdot 5^4} - \frac{4 \cdot 9 \cdot 14 \cdot 19 \cdot 24 \cdot 3^6}{5 \cdot 10 \cdot 15 \cdot 20 \cdot 25 \cdot 30 \cdot 5^6} - \&c.$$

Comparing this ſeries with N° 294. we get $b = \frac{3}{4}$, and $c = \frac{3}{4}$; hence, by ſubſtitution,

tution, and reducing, there arises

$$(\Sigma) = \frac{\sqrt[5]{\frac{5}{4}+\frac{3}{4}} + \sqrt[5]{\frac{5}{4}-\frac{3}{4}}}{2\sqrt[5]{\frac{5}{4}}}; \sqrt[5]{2}+\sqrt[5]{\tfrac{1}{2}} \text{ being a root of the equation}$$

$x^5 - 5x^3 - 5x = \frac{5}{2}$.

11. *Where the series to be summed is*

$$1 - \frac{6\cdot 4^2}{7\cdot 14\cdot 5^2} - \frac{6\cdot 13\cdot 20\cdot 4^4}{7\cdot 14\cdot 21\cdot 28\cdot 5^4} - \frac{6\cdot 13\cdot 20\cdot 27\cdot 36\cdot 4^6}{7\cdot 14\cdot 21\cdot 28\cdot 35\cdot 42\cdot 5^6} - \text{&c.}$$

By comparing this series with the same formula, we find $b = \frac{5}{3}$, $c = \frac{4}{3}$; and therefore by substitution, &c. there arises

$$(\Sigma) = \frac{\sqrt[7]{\frac{5}{3}+\frac{4}{3}} + \sqrt[7]{\frac{5}{3}-\frac{4}{3}}}{2\sqrt[7]{\frac{5}{3}}}; \sqrt[7]{3} + \sqrt[7]{\tfrac{1}{3}} \text{ being a root of the equation}$$

$x^7 - 7x^5 - 14x^3 - 7x = \frac{10}{3}$.

12. *Required the sum of the infinite series*

$$\frac{1}{2}\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}} - \frac{3}{2\cdot 4} + \frac{1}{3\cdot 4}\times\sqrt{\frac{26-15\sqrt{3}}{26+15\sqrt{3}}} + \frac{3\cdot 5}{2\cdot 4\cdot 6} + \frac{3}{3\cdot 2\cdot 6} + \frac{1}{5\cdot 6}\times\sqrt{\frac{362-209\sqrt{3}}{362+209\sqrt{3}}}$$
$$- \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} + \frac{3\cdot 5}{3\cdot 2\cdot 4\cdot 8} + \frac{3}{5\cdot 2\cdot 8} + \frac{1}{7\cdot 8}\times\sqrt{\frac{5042-2911\sqrt{3}}{5042+2911\sqrt{3}}} + \text{&c.}$$

Comparing this series with N° 327. we find $x = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$, which is evidently the tangent of the arc of 15° radius 1. Hence $a = \frac{1}{6}A$; and by substitution we have

$$(\Sigma) = \frac{\sqrt{2+\sqrt{3}}}{12}A + \frac{2+\sqrt{3}-2\sqrt{2+\sqrt{3}}}{2\sqrt{2-\sqrt{3}}},$$

the sum of the proposed series.

EMENDATIONS AND ADDITIONS.

Some Typographical Errors, and other Inaccuracies, having inadvertently escaped both in the Original and Tranſlation of Mr. LORGNA's Treatiſe, it is preſumed the following Remarks, &c. will be of Service to the Reader.

IN the enunciation of Prop. I. p. xi. for *Number of Terms* read *Number or Place of the Term.* For the former part of the remark in p. 10. ſubſtitute this: Since the fluential expreſſion $\frac{1}{q}\int \frac{x^{\frac{p}{q}}\,\dot{x}}{1-x}$ is ſuppoſed to be generated while x from 0 becomes 1, it is evident that the ſeries, and conſequently the expreſſion for its ſum, ought, when x is under the former circumſtance, to be alſo $= 0$. It is therefore to be underſtood, not only here, but in every propoſition, that each fluential expreſſion, when reduced, is to be corrected by taking $x = 0$, and ſubtracting the reſult: And by finally ſubſtituting 1 for x, that the expreſſion is properly adjuſted to the propoſed ſeries, which is then ſaid to be *perfectly integral*, that being ſuppoſed its whole or ultimate value; which is indeed actually ſo when the ſigns of the propoſed ſeries are poſitive, as that value of x manifeſtly exhibits the *whole* fluent of the expreſſion when the higher ſign takes place in the denominator.

P. 16. l. 9. for the *ſemicolon* write C. l. 5. from b. r. $-\int y\,\dot{x}$.

P. 22. l. 16. r. *an even number*. P. 24. l. 2. from b. r. L. $\overline{1+y}$.

P. 25. l. 15. for + L. $\overline{1-\sqrt{x}}$, r. + L. $\overline{1-\sqrt[3]{x}}$. l. 16. for L. 2. r. L. $\frac{4}{3}$. l. 17. for L. $\frac{2}{\sqrt{3}}$, r. L. $\frac{4}{3\sqrt{3}}$.

P. 28. l. 2. draw a vinculum from the firſt fluential ſign over the whole expreſſion. l. 8. for *latter fluential becomes*, r. *fluentials become*.

P. 29. l. ult. for $\frac{x}{\pi}$, r. $\frac{x^{\pi}}{\pi}$. P. 38. l. 2. r. $\beta = +1$; l. 8. for $-1\frac{1}{3}$, r. $+1\frac{1}{3}$; l. 11. and 12. for x, r. y; l. 19. for *root* r. *quantity*; l. 26. for $\frac{1}{2}\sqrt{3}\cdot\frac{1}{2}\sqrt{3}\cdot y$, r. $\sqrt{3}\cdot\frac{1}{2}\sqrt{3}\cdot y$. P. 44. l. 19. r. $\frac{1}{3\cdot 4\cdot 6}$. P. 46. l. 3. for + &c. r. − &c. P. 47.

l. 4.

l. 4. from b. for $-\frac{3}{3^2}$ L. 3. r. $-\frac{3^2}{3^2}$ L. 3. P. 48. l. 3. for $1-x$ in the denominator, r. $1+x$. P. ib. l. ult. r. $-\frac{17}{9}$.

The method of transforming the expreſſion in p. 56. may perhaps by ſome be thought tedious, as it is evidently reducible to $2\dot{y} \times \frac{y^6-y}{1-y^2}$, which by diviſion immediately becomes $2\dot{y} \times -y^4-y^2-1+\frac{1}{1+y}$; but this was done in order to ſhew the various tranſmutations an expreſſion may paſs through before it arrives at the moſt ſimple form, which is well known to be particularly uſeful in the finding of fluents; for which reaſon I have alſo in ſome other places adopted this method.

P. 58. l. 4. for $= \frac{3}{16}$, r. $= -\frac{3}{16}$; l. 8. r. $\beta = +1$. P. 60. l. 5. draw a vinculum from the rad. ſign in the num. P. 61. l. 7. for $-\frac{1}{2}y$, r. $-\frac{1}{2}y^2$. P. 63. l. 11. after *numbers* add——But, it muſt be obſerved, that this is underſtood of the expreſſion in general, no particular relation being ſuppoſed to obtain between the coefficients of the fluentials. For, it is obvious, that if we can aſſume, or diſcover, ſuch values of p, q, &c. as may by ſubſtitution make any of the coefficients with contrary ſigns become equal, if the difference of the exponents of the correſponding fluentials be a *whole* number when the ſigns of the propoſed ſeries are +, or an *even* number when they are alternately + and −, the algebraic ſum will be had from the ſame formula, though ſome of the values of π, ω, &c. be fractions in the former caſe, or odd numbers in the latter. The ſeries in N° 265. Caſe 2. and 3. immediately ariſe from the formula R in the above circumſtances; which are all the poſſible caſes in that theorem when x is $= 1$.

P. 67. l. 14. By an inadvertent omiſſion (in the original) of the coefficient ($\frac{1}{2}$) of the ſecond term, the latter part of this art. is rendered erroneous, let it therefore be read as follows:——We muſt therefore have recourſe to the œcumenical formula R (art. 47.), in which ſubſtituting the above values of p, q, &c. there ariſes $\int \frac{\overline{1-x^{\frac{1}{3}}} \cdot \dot{x}}{1-x} - \frac{1}{2}\int \frac{\overline{1-x^{-\frac{2}{3}}} \cdot x\dot{x}}{1-x} + 4\int \frac{\overline{1-x^{-\frac{1}{6}}} \cdot x^{\frac{1}{2}}\dot{x}}{1-x}$, expreſſing the ſum of the given ſeries. Now put $x^{\frac{1}{6}} = y$, and the laſt expreſſion is changed to $6\int \frac{\overline{1-y^2} \cdot y^3\dot{y}}{1-y^6} - 3\int \frac{\overline{y^4-1} \cdot y^7\dot{y}}{1-y^6} + 24\int \frac{\overline{y-1} \cdot y^7\dot{y}}{1-y^6}$, which by a proper reduction becomes $\frac{24}{7}y^7 - 8y^3 + \frac{1}{2}y^6 + 3\int \frac{y^5\dot{y}}{1-y^6} + 24\int \frac{y^2\dot{y}}{1-y^6} - 27\int \frac{y\dot{y}}{1-y^6} =$

$\frac{24}{7}y^7 - 8y^3 + \frac{1}{2}y^6 + (4\text{ L.}\overline{1+y+y^3+y^4} - 4\text{ L.}\overline{1+y+y^2} - \frac{11}{4}\text{ L.}\overline{1+y^2+y^4} =$

4 L.

$4\,\mathrm{L}.\sqrt{1+y^3}+4\,\mathrm{L}.\frac{1+y}{1+y+y^2}-\frac{11}{4}\,\mathrm{L}.\sqrt{1+y^2+y^4}=$ (becauſe $\frac{1+y}{1+y+y^2}$ is $=\frac{1+y^3}{1+y^2+y^4}$))

$8\,\mathrm{L}.\sqrt{1+y^3}-\frac{27}{4}\,\mathrm{L}.\sqrt{1+y^2+y^4}$, $-\frac{9}{2}\sqrt{3}$. circ. arc, ſin. $\frac{\sqrt{3}\cdot y^2}{2\sqrt{1+y^2+y^4}}$, $r.1.=$ (when $y=1$) $6+4\,\mathrm{L}.4-\frac{27}{4}\,\mathrm{L}.3-9$ circ. arc, rad. $\frac{1}{2}\sqrt{3}$, tang. $\frac{1}{2}$.

The ſame is to be obſerved in the note p. *63.

P. 71. l. 8. r. $\frac{1}{2}$ L. &c.; l. 10. r. $\frac{2}{\sqrt{2}}$ L. &c.; l. 13. r. $-\frac{4}{3a^2}$ L. &c.; l. 16. dele $\frac{8}{3a^2}$ L. a, l. 2. from b. for $2\sqrt{2}$ L. &c. r. $\sqrt{2}$. L, and make the next term +; l. ult. dele the firſt term, and write t. $\frac{3}{4+2^{\frac{2}{3}}}$; ſo will the ſum of the ſeries be (P. 72. l. 1. and 2.) $\sqrt{2}\,\mathrm{L}.\frac{\sqrt{2}+1}{\sqrt{2}-1}-\frac{1}{3}\sqrt{2}\mathrm{L}.\sqrt{4^{\frac{2}{3}}+2^{\frac{1}{3}}+1}+\frac{4^{\frac{2}{3}}}{3}\,\mathrm{L}.\sqrt{2^{\frac{1}{3}}-1}$ $-\frac{4}{3}$ circ. arc, rad. $\frac{\sqrt{3}}{4^{\frac{1}{3}}}$, t. $\frac{3}{4+4^{\frac{1}{3}}}$, (original).

P. *64. l. 4. from b. r. L. $\sqrt{1+y^{12}}$. P. *67. l. 7. for $\frac{p}{p}$, r. $\frac{p}{q}$; l. ult. for *Art.* 55. r. *in the preceding art.* P. 77. l. 11. r. M; l. 12. for $\int x^{p:q}+\dot{x}$, r. $\int x^{p:q+\pi}\,\dot{x}$. P. 79. l. 3. from b. for HEB, r. HBE; l. 2. from b. for EK, r. FK. P. 85. l. 4. from b. for $\frac{8}{15}$ r. $\frac{5}{6}$; l. ult. for $\frac{16}{15}$, r. $\frac{10}{6}$. P. 86. l. 2. r. $\frac{488}{450}-\frac{19}{15}\,\mathrm{L}.2$, (original). P. 96. l. 17. add——*in a pure algebraic expreſſion.*

P. 99. l. 3. from b. r. $\frac{452}{741}$; l. ult. r. $\frac{14}{13\cdot19}\times\int\frac{\dot{y}}{1+y^3}=\frac{14}{3\cdot13\cdot19}\times:\frac{\dot{y}}{1+y}$ $-\frac{y\dot{y}}{1-y+y^2}+\frac{2\dot{y}}{1-y+y^2}$, and in the latter part of the example for 13 in the denom. of the coefficient of the log. r. 39. and make the circ. arc affirm. and then l. 6. p. 100. will become $\frac{452}{741}-\frac{187}{247}$ arc, 45° rad. 1, $+\frac{14}{39\cdot19}\,\mathrm{L}.2+$ &c. (original).

P. 100. Ex. V. This example is rendered erroneous (in the original) by a miſtake in ſubſtituting the values of p, q, &c. whereby the coefficient of the third term of the fluential (l. 16.) is put down $-\frac{11}{16}$ inſtead of $+\frac{11}{3\cdot16}$, let l. 19. therefore be read $\frac{11}{24}\int y^2\,\dot{y}-\frac{11}{24}\int\dot{y}+\frac{5}{6}\int\frac{\dot{y}}{1+y^2}$; l. 21. r. $-\frac{11}{36}+\frac{5}{6}$ &c. and l. 24. for

+

$+\frac{11}{24}-$, r. $-\frac{11}{36}+\frac{1}{6}$ &c. P. 102. l. 5. r. and $\frac{3\dot{y}}{1+y^3}$ is evidently $=$ $\frac{\dot{y}}{1+y}-\frac{y\dot{y}}{1-y+y^2}+\frac{2\dot{y}}{1-y+y^2}$; l. 6. for *latter* r. *second*; l. 12. after the period, add, *To which adding* $\left(\frac{2\dot{y}}{1-y+y^2}=\right)+\frac{2}{3}$ circ. arc, rad. $\frac{1}{2}\sqrt{3}$, t. $\frac{3y}{4-2y}$, *we have* $\int\frac{3\dot{y}}{1+y^3}=\mathrm{L}\cdot\overline{1+y}-\frac{1}{2}\mathrm{L}\cdot\overline{1-y+y^2}+2$ circ. arc, rad. $\frac{\sqrt{3}}{2}$, t. $\frac{3y}{4-2y}$. And therefore &c.; l. 13. r. $\frac{14}{3\cdot13\cdot19}\mathrm{L}\cdot2+$ &c. P. 103. l. 1. for *forms*, r. *terms*. P. 104. l. 6. from b. r. $\frac{3\dot{y}}{1+y^3}$, and l. 5. from b. for $-\frac{2}{3}$, r. $+2$. P. 106. l. 12. for $\cdot\times\lambda$, r. $+\cdot\times\lambda$. P. 110. Ex. V. for 16, 18, &c. in the denominators, r. 18, 22, &c. P. 111. Ex. VI. By a miſtake (in the original) in ſubſtituting the values of p, q, &c. in the general formula, the coefficients in the expreſſion for the ſum of this ſeries are rendered erroneous; let l. 5. from b. therefore be written $\frac{x}{6}-\frac{107}{15}x^2+$ $\frac{136}{3}\int\frac{x^{\frac{3}{2}}\dot{x}}{1+\sqrt{x}}=-\frac{105}{12}+\frac{136}{3}\int\frac{x^{\frac{3}{2}}\dot{x}}{1+\sqrt{x}}$; l. 3. from b. for 110. r. 136, and for 220, r. 272, then will the ſum become $\frac{272}{3}\mathrm{L}\cdot2-\frac{2219}{36}$.

P. 136. In Order III. it may be neceſſary to obſerve that in (F) z denotes the number of terms, but in (§) z denotes the laſt term, or $\overline{m+zn}$. P. 137. l. 7. write the index r at the end of the vincula. P. 140. l. 7. for $b\,d-a\,\delta$, r. $b\,d+a\,\delta$. l. 2. from b. for $\mathrm{Z}\,z\,a+\frac{z^2-z}{2}d$, r. $\mathrm{Z}\times z\,a+\overline{\frac{z^2-z}{2}d}$, and for $+\zeta$, r. $-\zeta$. P. 158. l. 2. from b. r. $\frac{x\dot{\mathrm{S}}}{\dot{x}}$. P. 159. l. 4. and 5. from b. for *even*, *uneven*, r. *equal*, *unequal*. P. 160. l. 2. r. Order XXVI. Form 100. P. 163. l. 5. from b. for $q-1$ r. $q+1$. P. 169. l. 3. from b. r. $\frac{\pi^3}{2\cdot16}$. P. 170. l. ult. for $\frac{\pi}{30}$, r. $\frac{\pi^3}{30}$. P. 184. l. 6. for a, b, c (z) &c. r. $a\,b\,c\,(z)$. P. 188. l. 9. for 5635, r. 5365; l. 17. part of the type having been diſplaced in working off this ſheet ſeveral copies are faulty, it ſhould be $\frac{8\cdot\overline{\frac{3}{4}}^{\frac{3}{2}}}{3}+\frac{\overline{\frac{3}{4}}^{\frac{1}{2}}}{2}+$ &c. P. 192. l. 7. after *written* add *in*. P. 200. l. 2. in the numerators of the terms, r. $2x^9$, $2^2\cdot1\cdot2x^{11}$, $2^3\cdot1\cdot2\cdot3x^{\frac{13}{2}}$, &c.; l. 7. r. $\frac{161}{15}x^5$.

P. 209.

P. 209. l. 3. Mr. Simpſon is the author of the differential ſeries, from whence this theorem is derived. P. 215. l. 4. r. $\int \frac{\dot{x}}{1+x}$; l. 6. after *whatſoever*, add *if* x *be* $= 1$; l. ult. for $x^{-\frac{1}{2}} \int$, r. $x^{-\frac{1}{2}} \dot{x} \int$.

As it has been objected that the ſums of the ſeries N^os 229, 230, &c. are not eaſily attainable from the fluential expreſſions there exhibited, I ſhall therefore not only give the fluents of thoſe expreſſions, but alſo ſhew the inveſtigation of them; as the method is very extenſive, and applicable to the finding of a variety of other ſeries with their ſummations.

The ſummation of the ſeries N° 227. may be immediately deduced from the general one N° 108. or N° 111. where p, n, r are reſpectively $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ in the former, or 1, 3, 1 in the latter; in which alſo $v = m = 2$. Now multiply this ſeries and its ſum by $\dot{x}$, take the correct fluent, and multiply the whole by $\frac{x^{-\frac{1}{2}}}{2}$, and there ariſes (by Form 35.)

(N° 228.) $\frac{1}{4}x + \frac{1 \cdot 5}{4 \cdot 6}x^2 + \&c. = \frac{1}{3^x} \times \int \frac{\dot{x}}{\overline{1-x}^{\frac{1}{2}}} - \dot{x} = \frac{2 \cdot \overline{1-x}^{-\frac{1}{2}} - x - 2}{3^x}$.

Multiply the ſame ſeries (N° 227.) by $x^{\frac{1}{2}}\dot{x}$, take the fluent, and multiply the whole by $\frac{x^{-\frac{3}{2}}}{2}$, and we have (by Form 36.)

(N° 229.) $\frac{x}{5} + \frac{5x^2}{4 \cdot 7} + \frac{5 \cdot 7x^3}{4 \cdot 6 \cdot 9} + \&c. = \frac{1}{3x^{\frac{3}{2}}} \times : \frac{2x^{\frac{3}{2}}}{\sqrt{1-x}} - 2\text{A, s. } \sqrt{x}, - \frac{2}{3}x^{\frac{3}{2}}$.

The ſame ſeries being in like manner multiplied by $x^{\frac{3}{2}}\dot{x}$, the fluent taken, and multiplied by $\frac{x^{-\frac{5}{2}}}{2}$, we have (by Form 38.)

(N° 230.) $\frac{1}{7}x + \frac{5x^2}{4 \cdot 9} + \&c. = \frac{1}{3x^{\frac{5}{2}}} \times : \frac{3x^{\frac{1}{2}} - x^{\frac{3}{2}}}{\sqrt{1-x}} - 3\text{A, s. } \sqrt{x}, - \frac{2}{5}x^{\frac{5}{2}}$.

And thus we may proceed as far as we pleaſe, getting a new ſeries, and its ſum, at every operation.

Now, from the ſums of theſe three ſeries is obtained the ſum of N° 132. by means of the general one N° 106. where b, c, d, &c. are reſpectively 1, $\frac{5}{4}$, $\frac{5 \cdot 7}{4 \cdot 6}$, $\frac{5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8}$, &c. being the coefficients to the ſucceſſive powers of x in N° 227. Hence by ſubſtitution,

&c.

&c. we have $\frac{1}{24x^{\frac{5}{2}}}\int\dot{x}\int x^{-\frac{1}{2}}\dot{x}\int\frac{\dot{x}}{\overline{1-x}^{\frac{3}{2}}}-\dot{x}=\frac{1}{18x}\times\frac{2}{\sqrt{1-x}}-x-2-\frac{1}{12x^{\frac{3}{2}}}\times\frac{2x^{\frac{1}{2}}}{\sqrt{1-x}}-\frac{2}{3}x^{\frac{3}{2}}-2A+\frac{1}{36x^{\frac{5}{2}}}\times\frac{3x^{\frac{1}{2}}-x^{\frac{3}{2}}}{\sqrt{1-x}}-\frac{2}{5}x^{\frac{5}{2}}-3A$; the fluents being ſuppoſed to be generated while x from 0 attains any given value not greater than 1, A being (as before) the circ. arc, rad. 1. ſin. $\sqrt{x}$.

If we take x a root of the equation $\frac{1}{6}x^{-\frac{3}{2}}-\frac{1}{12}x^{-\frac{5}{2}}=0$, we ſhall evidently expunge the circular arc, and the ſum of the ſeries will then be expreſſed algebraically. Now it appears that $\frac{1}{2}$ is a root of this equation, conſequently by ſubſtitution in N° 132. we have

$$\frac{1}{2.4.5.7.2}+\frac{5}{2.4.6.7.9.2^2}+\frac{5.7}{2.4.6.8.9.11.2^3}+\frac{5.7.9}{2.4.6.8.10.11.13.2^4}+\&c.$$

$=\frac{1}{3}\sqrt{\frac{1}{2}}-\frac{7}{30}$. And by ſubſtituting other ſeries in N° 106. may the ſummations of a number of different ſeries be deduced, which will always become algebraical by taking the ultimate value of x ſuch as may exterminate the circular arc.

REMARKS *on Mr.* LANDEN'S *Obſervations on Converging Series.*

WHEN men of ſuperior learning, and diſtinguiſhed abilities, point out to the world, with candour and good-nature, the real errors and miſtakes of others, they certainly lay us under the higheſt obligations to them, and merit our warmeſt acknowledgements; as an impartial enquiry after truth, wherever it is to be found, is the great foundation of all uſeful knowledge. But when, under colour of ſuch animadverſions, we find the plain import of an author's words miſrepreſented and perverted, and trifling, uneſſential inaccuracies magnified into enormous errors, we have the ſtrongeſt reaſons to ſuſpect that they have been dictated by a mind prepoſſeſſed with prejudice, and actuated by diſingenuouſneſs; and are therefore ſo far from having their deſired effect, that they render the writers thereof ridiculous indeed to every intelligent and unbiaſſed reader. Whether Mr. Landen's *Obſervations*, with reſpect to Mr. Lorgna's Treatiſe, ſtand in the former or latter of theſe predicaments, muſt be left to the determination of the candid and ingenuous, when they have attentively conſidered the following pages.

To be as conciſe as poſſible in theſe Remarks, I ſhall confine myſelf chiefly to two principal objections to what Mr. Landen has advanced in his Obſervations; which are,

1. That Mr. Landen's theorems cannot poſſibly be deduced from what Mr. Simpſon has done in his *Diſſertations* without adopting the ſame principles, or method of procedure, as laid down in Mr. Lorgna's Treatiſe.

2. That the *criterion* exhibited by Mr. Lorgna of the poſſibility of the algebraic ſummation of this claſs of ſeries, is not invalidated, or ſhewn to be defective, by any thing Mr. Landen has ſaid on the ſubject; but that he has, in his obſervations, either miſunderſtood, or perverted, the ſenſe thereof.

Now, the firſt objection is very evident from p. 4. (Obſerv.) where the author tells us, that before the following theorems can be deduced it is *neceſſary* to obſerve that

$$\frac{x^{s-1}}{1-x} \text{ is } = -x^{r-1} - x^{r} - x^{r+1} - x^{r+2} \ (s-r) + \frac{x^{r-1}}{1-x}$$

and therefore when $s-r$ is a poſitive integer the ſeries will terminate, and the infinite

or

or logarithmic part of the expreſſion, to which the above equation refers, will be expunged, when the ſum of the coefficients is equal to 0; or, as it is there expreſſed, when the indefinite quantity (x) is a root of the equation formed by making the ſum of thoſe coefficients $= 0$. But is not this the very principle upon which Mr. Lorgna inveſtigates his *formulæ* for the algebraic ſummation? For from this gentleman's words we underſtand, that when the ſum of a ſeries is expreſſed thus $Ax^{s-r}\int\frac{x^r\dot{x}}{1-x} - B\int\frac{x^s\dot{x}}{1-x}$ (uſing Mr. Landen's literal ſymbols) it will be attainable in finite or algebraic terms whenever $s-r$ is a (poſitive) whole number, and the value of x ſuch that the coefficients of the fluents may be equal (which, in this caſe, will always happen when x is $= 1$). And the reaſon aſſigned is, becauſe, in theſe circumſtances, the above expreſſion is equal $A\times\overline{\int\frac{x^r\dot{x}}{1-x} - \int\frac{x^s\dot{x}}{1-x}} = A\times\int\frac{\overline{x^r-x^s}\cdot\dot{x}}{1-x}$ $= A\times\int\frac{\overline{1-x^{s-r}}}{1-x}\times x^r\dot{x}$, and $\frac{1-x^{s-r}}{1-x}$ is $= 1+x+x^2+x^3$ $(s-r)$; which are the very requiſites premiſed by Mr. Landen for the algebraic ſummation, this laſt equation being evidently the ſame with that given above, which is there deſignedly rendered obſcure by an unneceſſary and awkward tranſpoſition.

I ſhall here juſt obſerve to the reader, that I am ſenſible, it would ill become me, were I ſo diſpoſed, to paſs any cenſure on the valuable works of ſo eminent, worthy, and reſpectable a character as was the late Mr. Thomas Simpſon; and would undoubtedly betray a moſt ungenerous and unſcientific mind to offer to *under-rate* or *miſ-repreſent* them. But, on this occaſion, I think I may with truth take the liberty to ſay, that it does not appear from the inveſtigations of Mr. Simpſon's theorems (p. 62. Diſſert.), that he had any idea of the method given by Mr. Lorgna reſpecting the *algebraic* ſummation of a ſeries *, as other principles were wanting for this purpoſe which Mr. S. had not premiſed; and therefore, that circumſtance muſt, by his proceſs, be merely adventitious. Neither does it appear that Mr. Simpſon even foreſaw the uſe which might be made of the propoſition p. 146. Diſſert. in the ſumming of ſeries; the whole purport of that prop. being evidently only to reduce a compound fluxional expreſſion into ſingle terms, in order to facilitate the finding of the fluent thereof. And, in all probability, had not Mr. Lorgna publiſhed his treatiſe on *Converging Series*, we ſhould never have ſeen Mr. Simpſon's theorems put to the rack, in order

* This, indeed, Mr. Emerſon has given a ſmall ſpecimen of in his Fluxions, p. 158. where he ſhews under what conditions the expreſſion $\overline{\int x^n\dot{x}\int\frac{\dot{x}}{1-x}}$ will become algebraical; and thereby ſums the ſeries ariſing from expanding this expreſſion, and taking the fluents of the terms. The formula N° 286. is deduced from hence.

to extort a ſecret from them, which, it plainly enough appears, they do not contain. It muſt, however, in juſtice to Mr. Simpſon, be acknowledged, that the above-mentioned propoſition is eſſentially the ſame (only not ſo general in ſome reſpects) as that which Mr. Lorgna has made the foundation of his method of ſummation; but the ſuperſtructure he has raiſed thereon has not the leaſt ſimilarity to any thing Mr. S. has done*. And this is all Mr. Landen can, with veracity, aſcribe to Mr. S. in the

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* Whoever will take the trouble to compare Mr. Lorgna's theorems with Mr. Simpſon's prop. IV. and V. in his Diſſertations (theſe being the only prop. in that treatiſe which relate to the ſummation of this claſs of ſeries) will find that they are intirely different from each other. Mr. S. in the former prop. exhibits the ſum of the ſeries there propoſed by means of another aſſumed ſeries, whoſe ſum being given is denoted by S. This he effects by reſolving the terms of the propoſed ſeries into as many ſimple fractions as there are factors in the denominators of thoſe terms by the foregoing lemma, the firſt, ſecond, third, &c. terms of which reſpectively conſtitute other ſeries with a common coefficient to the terms, for each of which he obtains a finite expreſſion by ſubtracting as many initial terms of the aſſumed or given ſeries from its ſum S as are found to precede the initial term of the ſeveral ſeries having a common coefficient, that is, till a term of S coincides with or becomes $\frac{z^{p-n}}{p-n}$, $\frac{z^{q-n}}{q-n}$, &c. which can happen only when the values of p, q, m, &c. are ſo related that $m+zn$ may be equal $p-n$, $q-n$, &c. or, $\frac{p-m}{n}$, &c. $= z$ [illegible] which muſt evidently be a whole number ſince z is the index of the number of terms leſs 1 of the ſeries S. And the proceſs in prop. V. is evidently by reſolving the compound coefficients of the terms of the propoſed ſeries each into a number of ſimple ones by the ſaid lemma, and then ſuppoſing the ſums of the ſeveral ſeries formed of the terms which have the ſame coefficient to be each *given*. But what ſimilarity has all this to Mr. Lorgna's proceſs? Had Mr. S. proceeded on Mr. Lorgna's principles, when he had brought his ſeries (prop. IV.) to this form

$$\frac{A}{z^p}\times:\frac{z^p}{p}\pm\frac{z^{p+n}}{p+n}+\&c.$$

$$\frac{B}{z^q}\times:\frac{z^q}{q}\pm\frac{z^{q+n}}{q+n}+\&c.$$

&c.

inſtead of introducing the ſeries $\frac{z^m}{m}\pm\frac{z^{m+n}}{m+n}+$ &c. whoſe ſum muſt neceſſarily be given, he would have expreſſed it thus

$$\frac{A}{z^p}\int\frac{z^{p-1}\dot{z}}{1\mp z^n}\mp\frac{B}{z^q}\int\frac{z^{q-1}\dot{z}}{1\mp z^n}+\&c.$$

which is a general expreſſion for the ſum of the ſeries, whatever the values of a, b, &c. p, q, &c. z, are. And the requiſites for the algebraic ſummation are (ſuppoſing, for example, only two terms to be taken, and z^n put $= x$) that $\frac{A}{x^{\frac{p}{n}}}$ may $= \frac{B}{x^{\frac{q}{n}}}$, and $\frac{q-p}{n}$ be a *whole* (poſitive) number, an *even* number, or an *odd* number, according to the nature of the ſigns. That is, if the ſigns be all $-$, $\frac{q-p}{n}$ muſt be a *whole* number; if all $+$, an

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the inveſtigations of his theorems; the whole of what he has exhibited beſide in his *Obſervations* reſpecting the ſummation of this claſs of ſeries, being indiſputably Mr. Lorgna's.

It is then very evident from what has been ſaid above, that the theorems given in Mr. Landen's Obſervations are inveſtigated on the ſame principles, and the ſame conditions we ſee are acknowledged to be abſolutely requiſite for the algebraic ſumma-

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odd number; if — B and the other ſigns +, an *even* number. In other caſes, the ſum of the ſeries will be infinite, or expreſſed only by circular arcs, or logarithms. For (in the caſe of two fluentials only) when the coefficients are equal, and x^n made $= x$, the above expreſſion becomes

$$\frac{A}{\frac{p}{n}x^{n}} \times \int \frac{x^{\frac{p-n}{n}}\dot{x}}{1 \mp x} \mp \int \frac{x^{\frac{q-n}{n}}\dot{x}}{1 \mp x}, \text{ or } \frac{A}{\frac{p}{n}x^{n}} \int \frac{x^{\frac{q-p}{n}}}{1 \mp x} \times x^{\frac{p-n}{n}}\dot{x},$$

which will *always* be an algebraic quantity in the above-mentioned circumſtances; the fluents being ſuppoſed to be generated while x from o attains the value ariſing from equating the coefficients of the fluentials. We do not find any thing ſimilar to this either expreſſed or implied in the writings of Mr. Simpſon.

Now the properties of theſe two Gentlemen's theorems are very obvious. Mr. Simpſon's theorems (prop. IV. and V.) cannot become algebraic expreſſions unleſs S be exterminated, and this can only be effected in a general way, when there are *two* more factors (at leaſt) in the denominators of the terms of the ſeries than in the numerators, and the ultimate value of x is 1, which appears from coroll. 3. p. 82. where the expreſſion for the ſum in that caſe becomes a pure algebraic quantity. From whence we ſhould naturally be led to conclude, that when a propoſed ſeries does not come within theſe limitations, the ſum is expreſſible only by circular arcs, &c. And in other caſes it is obſervable, that we are wholly uncertain reſpecting the algebraic ſummation, though the relation of the factors come under the conditions ſpecified; this being purely accidental, as has been before obſerved. And when it happens that the ſum of a ſeries is not attainable algebraically by theſe theorems, that is, when the coefficient of S is ſuch that we cannot diſcover any value for x whereby it may be exterminated, there is little or no advantage in introducing the ſeries S, unleſs we are furniſhed with a number of fluents that may ſuit any caſe propoſed by the different values of m and n; without this it would be equally as expeditious to find the fluents of the original fluentials themſelves as they ariſe in the theorem. And here it may be obſerved, that when $\frac{p-m}{n}$, $\frac{q-m}{n}$, &c. happen to be fractional or negative numbers, that is, if $p-n$, $q-n$, &c. be o, or negative, S is no longer $= \int \frac{x^{m-1}\dot{x}}{1 \mp x^{n}}$, but is of different values in the theorem, *viz.* $\int \frac{x^{p-1}\dot{x}}{1 \mp x^{n}}$, $\int \frac{x^{q-1}\dot{x}}{1 \mp x^{n}}$, &c. though this neceſſary remark is omitted in Mr. Simpſon's note to prop. IV. And laſtly, in theſe theorems, it is *always* neceſſary that the factors in the terms of the propoſed ſeries be in the *ſame* arith. progreſſion, or be reduced to ſuch before the ſum can be exhibited thereby. But Mr. Lorgna's theorems are general for this claſs of ſeries, no reſtrictions whatever being neceſſary for the *geometric* ſummation,——the ſums of no given ſeries being previouſly requiſite,——have the neceſſary conditions pointed out for the *algebraic* ſummation; and that when the number of factors in the numerators are only *one* leſs than the number of thoſe in the denominators,——and laſtly, are immediately adapted to ſeries whoſe factors are in the ſame, or each in a different arith. progreſſion.

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tion, as thoſe laid down by Mr. Lorgna. And it will be equally as evident to any one that will compare the *formulæ* in the beginning of this Supplement with thoſe in the above-mentioned obſervations, that the latter are particular caſes of the former; and therefore, in ſuch caſes, the theorems are *actually the ſame*. To put this matter, however, beyond a doubt, let us take Mr. Lorgna's theorem, p. 17, &c.

$$\frac{1}{nq\cdot\frac{m}{n}-\frac{p}{q}}\times x^{\frac{m}{n}-\frac{p}{q}}\int\frac{x^{\frac{p}{q}}\dot{x}}{1-x}-\frac{1}{nq\cdot\frac{m}{n}-\frac{p}{q}}\int\frac{x^{\frac{m}{n}}\dot{x}}{1-x};$$ which expreſſion

by putting $A=\frac{1}{nq\cdot\frac{m}{n}-\frac{p}{q}}$, and $B=-\frac{1}{nq\cdot\frac{m}{n}-\frac{p}{q}}$, becomes

$$X^{\frac{m}{n}}\int\frac{\dot{x}}{1-x}\times\frac{Ax^{\frac{p}{q}}}{X^{\frac{p}{q}}}+\frac{Bx^{\frac{m}{n}}}{X^{\frac{m}{n}}},$$ the fluents being generated while x from 0 attains the value denoted by X. Now this expreſſion is evidently equal to

$$X^{\frac{m}{n}}\times\int Ax^{\frac{p}{q}}.X^{-\frac{p}{q}}+Bx^{\frac{p}{q}}.X^{-\frac{m}{n}}\times\frac{\dot{x}}{1-x}-Bx^{\frac{p}{q}}.X^{-\frac{m}{n}}-Bx^{\frac{m}{n}}.X^{-\frac{m}{n}}\times\frac{\dot{x}}{1-x},$$

that is, to

$$X^{\frac{m}{n}}\times AX^{-\frac{p}{q}}+BX^{-\frac{m}{n}}\times\int\frac{x^{\frac{p}{q}}\dot{x}}{1-x}-BX^{-\frac{m}{n}}\int\frac{x^{\frac{p}{q}}-x^{\frac{m}{n}}}{1-x}\dot{x};$$ but when

$\frac{m}{n}-\frac{p}{q}$ is a poſitive whole number, $\frac{x^{\frac{p}{q}}-x^{\frac{m}{n}}}{1-x}$ is $=x^{\frac{p}{q}}+x^{\frac{p}{q}+1}+x^{\frac{p}{q}+2}\left(\frac{m}{n}-\frac{p}{q}\right)$,

and conſequently the above expreſſion then becomes

$$X^{\frac{m}{n}}\times AX^{-\frac{p}{q}}+BX^{-\frac{m}{n}}\times\int\frac{x^{\frac{p}{q}}\dot{x}}{1-x}-BX^{\frac{p}{q}-\frac{m}{n}+1}\,q\times:\frac{1}{p+q}+\frac{X}{p+2q}+\frac{X^2}{p+3q}\left(\frac{m}{n}-\frac{p}{q}\right).$$

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By the ſame proceſs is the theorem p. 41. changed into

$$X^{\frac{r}{s}} \times : AX^{-\frac{p}{q}} + BX^{-\frac{m}{n}} + CX^{-\frac{r}{s}} \times \int \frac{x^{\frac{p}{q}}\dot{x}}{1-x} + qX^{\frac{r}{s}+1}$$

$$\times \left\{ \begin{array}{l} -B\dfrac{1}{X^{\frac{m}{n}-\frac{p}{q}}} \times \dfrac{1}{p+q} + \dfrac{X}{p+2q} + \dfrac{X^2}{p+3q}\left(\dfrac{m}{n} - \dfrac{p}{q}\right) \\ -C\dfrac{1}{X^{\frac{r}{s}-\frac{p}{q}}} \times \dfrac{1}{p+q} + \&c. \qquad \left(\dfrac{r}{s} - \dfrac{p}{q}\right) \end{array} \right.$$

And in like manner may we proceed with the ſubſequent theorems, till at laſt we obtain by induction the general theorem p. 7. Supp.

Now, in the lemma (p. 16. Series) it is obvious that y may denote any algebraic expreſſion whatſoever in terms of x and known quantities; therefore, if we take $y = \frac{x^{\frac{p}{q}}}{\overline{1-bx}^{k}}$, Mr. Lorgna's theorem p. 17. becomes

$$\frac{x^{\frac{m}{n}-\frac{p}{q}}}{nq.\frac{m}{n}-\frac{p}{q}} \int \frac{x^{\frac{p}{q}}\dot{x}}{\overline{1-bx}^{k}} - \frac{1}{nq.\frac{m}{n}-\frac{p}{q}} \int \frac{x^{\frac{m}{n}}\dot{x}}{\overline{1-bx}^{k}},$$

and the ſeries correſponding (expanding $\overline{1-bx}^{k}$ by means of the binomial theorem) is

$$\frac{x^{\frac{m}{n}+1}}{p+q \cdot m+n} - \frac{kbx^{\frac{m}{n}+2}}{p+2q.m+2n} + \frac{k.\frac{k-b}{2}b^2x^{\frac{m}{n}+3}}{p+3q \cdot m+3n} - \&c.;$$

and the laſt expreſſion but one is evidently equal to

$$\frac{1}{nq} \times X^{\frac{m}{n}} \times : AX^{-\frac{p}{q}} \int \overline{1-bx}^{k} x^{\frac{p}{q}}\dot{x} + BX^{-\frac{m}{n}} \int \overline{1-bx}^{k} x^{\frac{m}{n}}\dot{x}, \text{ where}$$

$A = \frac{1}{\frac{m}{n}-\frac{p}{q}}$, $B = -\frac{1}{\frac{m}{n}-\frac{p}{q}}$, and X = the ultimate value of x, as before.

But, it appears that the latter fluential $\int \overline{1-bx}^{k} x^{\frac{m}{n}}\dot{x}$ is $= \int \overline{1-bx}^{k} x^{\frac{p}{q}+\pi}\dot{x}$, and therefore when $\pi\left(\frac{m}{n} - \frac{p}{q}\right)$ is a poſitive whole number the fluent (by Mr. Emer-

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ſon's 11th form) will be

$$\frac{\frac{p}{q}+\pi\cdot\frac{p}{q}+\pi-1\cdot\frac{p}{q}+\pi-2\,(\pi)}{\frac{p}{q}+k+\pi+1\cdot\frac{p}{q}+k+\pi\cdot\frac{p}{q}+k-1\,(\pi)\,b^{\pi}}\times\int\overline{1-bx}^{k}\,x^{\frac{p}{q}}\,\dot{x}$$

$$-\frac{x^{\frac{p}{q}+\pi}}{\frac{p}{q}+k+\pi+1.b}\times\overline{1-bx}^{k+1}+\frac{\frac{p}{q}+\pi\cdot A}{\frac{p}{q}+k+\pi\cdot bx}+\frac{\frac{p}{q}+\pi-1\cdot B}{\frac{p}{q}+k+\pi-1\cdot bx}\,(\pi),$$

which (by writing any whole number for π) is evidently $=$

$$\frac{\frac{p}{q}+1.\frac{p}{q}+2\,(\pi)}{k+\frac{p}{q}+2.k+\frac{p}{q}+3\,(\pi)\,b^{\pi}}\times:\int\overline{1-bx}^{k}\,x^{\frac{p}{q}}\,\dot{x}-x^{\frac{p}{q}+1}\times\overline{1-bx}^{k+1}\times:\frac{1}{\frac{p}{q}+1}$$

$$+\frac{k+\frac{p}{q}+2\cdot bx}{\frac{p}{q}+1\cdot\frac{p}{q}+2}+\frac{k+\frac{p}{q}+2\cdot k+\frac{p}{q}+3\cdot bx^{2}}{\frac{p}{q}+1\cdot\frac{p}{q}+2\cdot\frac{p}{q}+3}\,(\pi)\,;$$

and conſequently the laſt expreſſion (in this caſe) becomes

$$\frac{1}{nq}X^{\frac{m}{n}}\times:AX^{-\frac{p}{q}}+\frac{\frac{p}{q}+1\cdot\frac{p}{q}+2\,(\pi)\,BX^{-\frac{m}{n}-1}}{k+\frac{p}{q}+2.k+\frac{p}{q}+3\cdot(\pi)\,b^{\pi}}\times\int\overline{1-bx}^{k}\,x^{\frac{p}{q}}\,\dot{x}$$

$$-\frac{\frac{p}{q}+1\cdot\frac{p}{q}+2\,(\pi)\,B\cdot\overline{1-bX}^{k+1}}{k+\frac{p}{q}+2\cdot k+\frac{p}{q}+3\,(\pi)\,b^{\pi}X^{\pi-1}}\times:\frac{1}{\frac{p}{q}+1}+\frac{k+\frac{p}{q}+2\cdot bX}{\frac{p}{q}+1\cdot\frac{p}{q}+2}\,(\pi).$$

By the ſame procedure is deduced the theorem in p. 8. Supp. Some other fluxionary expreſſions may alſo be diſcovered whoſe fluents have the form requiſite to the conſtruction of theorems ſimilar to the above, by means of Mr. Emerſon's forms 11, 12, 13, 14; or by the propoſitions in Mr. Simpſon's Diſſert. p. 149, &c.

Thus we ſee that theſe "*narrow, embarraſſed, ill-expreſſed, particular formulæ*" do, by a moſt eaſy and obvious tranſmutation, exhibit theorems evidently more comprehenſive than the *all-comprizing Landenian-Simpſonian* theorems, which have been poſitively aſſerted by Mr. Landen to contain at leaſt *all* the formulæ in Mr. Lorgna's work (excepting thoſe in the IX Sect.), when, in fact, they do not comprehend one

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ſingle theorem in that treatiſe; even the general expreſſions Mr. Landen has exhibited in his obſervations for the ſummations of theſe ſeries, being (as there expreſſed) only particular * caſes of the above formulæ, which, it is obvious, immediately reſult from thoſe given by Mr. Lorgna.

The ſecond objection I have to make to Mr. Landen's Obſervations is, that the *criterion* of the algebraic ſummation of this claſs of ſeries exhibited by Mr. Lorgna, is therein groſsly miſrepreſented; and therefore, that the Obſervator muſt either have miſunderſtood, or deſignedly perverted, the ſenſe thereof.

Now the plain import of Mr. Lorgna's words are, 1. That if there be two fluential expreſſions as $A\int \frac{x^{\frac{p}{q}}\dot{x}}{1\mp x} = B\int \frac{x^{\frac{m}{n}}\dot{x}}{1\mp x}$, whoſe coefficients A and B are equal to each other, and $\frac{m}{n} - \frac{p}{q}$ (π) a *whole* (poſitive) number, the whole expreſſion will be reducible into finite or algebraic terms, ſuppoſing the higher ſigns to take place. 2. If A be $=$ B, and the lower ſign be uſed in the denominators *only*, the whole expreſſion will be finite when π is an *even* number. 3. If A $=$ B, and the lower ſigns be uſed, the ſame expreſſion will be finite when π is an *odd* number. 4. And if the higher ſign be taken in the denominators, and the lower between the fluentials, the whole expreſſion will be infinite, whatever be the values of A, B, and π. Now this is all very evident by reducing the above expreſſion into $A\int \frac{1\mp x^{\frac{m}{n}-\frac{p}{q}}}{1\mp x} \times x^{\frac{p}{q}}\dot{x}$, from whence we may eaſily be convinced by diviſion, that $\frac{1-x^{\pi}}{1-x}$ will terminate in π terms when π is any *whole* number, $\frac{1-x^{\pi}}{1+x}$ in π terms

* If, in Mr. Lorgna's theorems, we take b, e, &c. q, n, s, &c. each $=$ 1, and a, c, &c. p, m, r, &c. $=$ to A—1, C—1, &c. P—1, M—1, R—1, &c. reſpectively (ſuppoſing P, M, R, &c. to be the factors in the denominators of the terms of the ſeries, and A, C, &c. thoſe in the numerators) we reduce them to the very ſame forms given by Mr. Landen, which are therefore evidently particular caſes only of Mr. Lorgna's theorems. The peculiar advantage attending this laſt gentleman's method is, that the theorems are immediately adapted to ſeries whoſe factors are in the ſame or *any other* arith. progreſſion; whereas thoſe given by Mr. Landen are reſtricted to the *ſame* progreſſion, as well in the numerators as denominators, and we do not obtain the ſum of a ſeries from thence till we have reduced each of the factors to the ſame progreſſion, by changing ſome, or perhaps all, of them into fractional numbers, which in various inſtances will be attended with difficulty and awkward ſubſtitutions.——Is this then the *extenſiveneſs* and *ſuperior elegance* which Mr. Landen would perſuade us is to be found in *his* method of expreſſing theſe theorems? O! Candour, Candour, Where doſt thou hide thyſelf?

when it is an *even* number, $\frac{1+x^{\pi}}{1+x}$ in π terms when it is an *odd* number, and that $\frac{1+x^{\pi}}{1-x}$ will not terminate, but run on *fine fine* in a ſeries, whatever the value of π may be. And as by this method of ſummation the theorems are always exhibited by a combination or compariſon of each fluential with that formed from the *laſt* factor in the denominator of the ſeries, we may thence eaſily deduce all the poſſible caſes of the algebraic ſummation, and the relation of the factors requiſite thereto. For inſtance, in N° 265 (which is the ſame with that in order 14.) if we equate the coefficient of the firſt fluential with that of the fifth, we ſhall evidently expunge the ſecond and ſixth terms, and conſequently if ω and $\pi-\delta$ be whole numbers, the expreſſion (Σ) will be finite whatever the values of π and δ may be ſeparately, provided that x be taken of ſuch value as to make the ſum of the coefficients become $= 0$. Again, if we equate the coefficients of the third and fifth fluentials, and π and $\omega-\delta$ be whole numbers, the expreſſion (Σ) will be finite, let ω and δ be whatever they may ſeparately, provided x be taken as above *. And thus, by combining or comparing the coefficients of the *laſt* fluentials with different ſigns, may all the poſſible caſes in any ſeries of this claſs be immediately diſcovered when the ſummation will be had algebraically, and literal theorems will ariſe by ſubſtitution in the reſpective general formulæ in all ſuch caſes; which is done in the ſeries referred to above (N° 265.). Now, if in this literal ſeries, we aſſume $p = b-e$, $q = 1$, $m = b$, $n = 1$, $r = 2b-g-2e$, $s = 2$, $t = 2b+g$, and $u = 2$; and ſubſtitute theſe values in the expreſſion (Σ) of the firſt particular caſe, there will ariſe the ſeries N° 272. and the expreſſion for its ſum. In which ſeries if we take $g = 1$, and $e = 1$, we

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* And to theſe caſes it is a very eaſy matter to adapt any number of ſeries at pleaſure. For in the former caſe, let $\pi-\delta = e$ a whole number, then will $\omega = e+2\delta$, and therefore $\delta = \frac{\omega-e}{2}$, $\pi = \frac{\omega+e}{2}$, from whence by aſſuming different values for e and ω, and taking p, q, &c. ſuch as may correſpond to thoſe values, we get as many ſeries as we pleaſe, that are exceptions to the general rule, and are yet ſummible by the ſame theorem. As for inſtance, let $e = 1$, and $\omega = 2$, then is $\delta = \frac{1}{2}$, and $\pi = \frac{3}{2}$. If $e = 1$, and $\omega = 4$, then is $\delta = \frac{3}{2}$, and $\pi = \frac{5}{2}$. If $e = 2$, and $\omega = 3$, then is $\delta = \frac{1}{2}$, and $\pi = \frac{5}{2}$, &c. If the ſigns of the propoſed ſeries be $+$, and $-$; let $\omega = 8$, and $e = 2$, then is $\delta = 3$, $\pi = 5$. If $\omega = 6$, and $e = 4$, then is $\delta = 1$, and $\pi = 5$, &c. In the latter caſe; let $\omega-\delta = e$ a whole number, then will $\omega = \frac{\pi+e}{2}$, and $\delta = \frac{\pi-e}{2}$; and by taking e and π of certain values, we derive as before other ſeries adapted to theſe caſes. Let (for inſtance) $\pi = 2$, $e = 1$, then is $\delta = \frac{1}{2}$, $\omega = \frac{3}{2}$. &c. If $\pi = 4$, $e = 2$, then is $\delta = 1$, $\omega = 3$. &c. In all theſe inſtances will the ſums of the ſeries be attainable in finite terms, though π, δ, &c. be fractions, the ſigns being $+$; or *odd* numbers, when the ſigns are $+$ and $-$.

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obtain Mr. Landen's theorem p. 28. Obſerv. which therefore appears to be only a particular caſe of that general theorem. Again, if we aſſume $p = b-e+1$, $q = 1$, $m = 2b+g+2$, $n = 2$, $r = 2b+g-2e+2$, $s = 2$, $t = b+g+1$, $u = 1$; and write theſe values in the ſame ſeries, and alſo in the expreſſion (Σ) of the ſecond particular caſe, there will reſult the ſeries N° 273. and its ſum in finite terms. And in this ſeries, if we take $g = 2$, $e = 2$, and multiply the whole by 4 (obſerving the remark when $\omega-\delta$ is negative) there ariſes Mr. Landen's theorem p. 12. Obſerv. which alſo appears to be a particular caſe only of the ſeries N° 273. the expreſſion for the ſum firſt becoming by ſubſtitution $\frac{1}{48} \times \overline{\frac{1}{b} - \frac{1}{b+1} + \frac{1}{b+2} - \frac{1}{b+3}} - \frac{1}{12} \times \overline{\frac{1}{2b+2} - \frac{1}{2b+4}}$; and by reducing, and multiplying by 4,

$$\tfrac{1}{2} \times \frac{1}{b.b+1.b+2.b+3}.$$

Here then we ſee plainly the grounds of Mr. Lorgna's algebraic ſummation of this claſs of ſeries. For the expreſſion in p. 63. which Mr. Landen has laid ſuch ſtreſs upon, evidently relates only to the general theorem when there is no given relation ſuppoſed between the factors; nothing being more obvious than Mr. Lorgna's knowledge of the algebraic ſummation depending on the different combinations of the fluentials in the general formulæ, by his exhibiting the particular caſes when the theorem in p. 96. is algebraically ſummible; and his not particularizing thoſe caſes in ſome of the other theorems, can, at the worſt, be called only an inadvertent omiſſion. Now, if Mr. Landen can make it appear, that the fluent of the binomial expreſſion before-mentioned can be had or exhibited in *algebraic* terms (that is, excluſive of circular arcs or logarithms) in any other circumſtances than thoſe we have enumerated, then will Mr. Lorgna's *criterion* fall to the ground; for upon no other principle can it be invalidated or ſhewn to be defective. But what ſhall we ſay when we find Mr. Landen himſelf expreſsly acknowledging the impoſſibility of it: For, in his firſt corollary, he tells us, that the expreſſion $\frac{1}{s-r} \times \int \frac{\dot{x}}{1 \mp x} \times \overline{\frac{x^{r-1}}{a^r} - \frac{x^{s-1}}{a^s}}$ (which is evidently $= A a^{s-r} \int \frac{x^{r-1}\dot{x}}{1 \mp x} - B \int \frac{x^{s-1}\dot{x}}{1 \mp x}$) will be *algebraical*, when $s-r$ (that is $\overline{s-1}-\overline{r-1}$) is a *whole* number, taking the higher ſigns; or an *even* number, uſing the lower ſigns; *but in other caſes the fluent will be aſſigned by circular arcs and logarithms.* Here then has Mr. Landen come to the ſame point with Mr. Lorgna; having, we have before ſhewn, inveſtigated his theorems for the algebraic ſummation (or rather given us the ſame theorems again, a little diſguiſed), and *now*

pointed out the poſſibility of ſuch ſummation, *both* on the very ſame principles with Mr. Lorgna! Now, would not any impartial perſon enquire with aſtoniſhment,—what could be Mr. Landen's motive in publiſhing thoſe *obſervations?* Was it to ſhew the fallacy of the *criterion* above-mentioned, he has entirely defeated his own purpoſe by miſtaking the principles; what he has ſaid thereon being, to the laſt degree, futile, and *frivolous* indeed; and, in ſhort, from what has been ſaid above, is evidently nothing at all to the purpoſe. Or was it, invidiouſly to pluck the laurels from the head of a man who appears to have *fairly* merited them, by publicly miſrepreſenting and defaming his work,———this deſign is alſo rendered abortive: For the miſtakes which Mr. Landen (and ſome others) have found in Mr. Lorgna's treatiſe, are evident to every one not miſtakes in principle or eſſential errors, but only trifling inaccuracies, or negligencies*, in the application of ſome few examples to the general theorems; which theorems are *now* univerſally acknowledged to be "*exceedingly accurate, extenſive, and ingenious.*" And with regard to the *Commentator*, I muſt take the liberty to inform the Obſervator, as it may be of ſervice in future, that illiberal language is not argument; and that the epithets *oſtentatious, booſting, vaunting*, &c. which he has ſo profuſely heaped upon him, appear to be ill-beſtowed on the man whoſe character is *publicly* known to be juſt the reverſe†.

Should Mr. Landen be diſpoſed to make any reply to what is advanced in theſe remarks, or to vindicate what he has already ſaid on the ſubject, it may not be improper to inform him, that it will be expected he firſt of all ſhew the poſſibility of obtaining the fluent of the binomial expreſſion ſo often mentioned, in finite or algebraic terms, in any other caſe beſide thoſe pointed out by Mr. Lorgna, and to confirm the ſame by examples. It is alſo wiſhed he would give the reaſon why he has omitted Mr. Lorgna's VII. Section; as alſo why he has not ſummed one ſingle ſeries *algebraically* by his theorems, when the number of factors in the numerators is only *one leſs* than the number of thoſe in the denominators, the ſigns being alternately + and −, and $x = 1$; or, as Mr. Landen expreſſes it, when x is $= -1$. But Mr. Landen ſays (p. 21. 22. Obſerv.) that, in the above circumſtance of the factors, it is *well known* when the ſigns are +, the ſum of ſuch ſeries will be infinite; but if + and − alternately, the ſum will be had in finite (algebraic) terms, x being $= 1$. Now this is exceedingly obvious from Mr. Lorgna's method of inveſtigation; the re-

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* To this cauſe we may certainly attribute the overſight in the expreſſion L. $\frac{1-y}{1-y^3}$; for it cannot be ſuppoſed that a perſon who could inveſtigate the general theorems in that treatiſe, which are carried through ſuch a number of intricate tranſformations, could be ignorant of ſo trifling a matter, as $\frac{1-y}{1-y^3}$ being $=$ to $\frac{1}{1+y+y^2}$.

† *Vid.* Hiſt. of Manchester, Oct. Edit. Vol. II.

quiſites

quiſites in the latter caſe being eaſily diſcovered by equating the coefficients of the fluentials in the correſponding theorems. But as Mr. L. ſays theſe things are *well known*, it is to be wiſhed that he would inform us, what Author beſides Mr. Lorgna hath ever ſhewn in a *general manner* the impoſſibility of the ſummation in the former caſe, and the conditions or neceſſary relation of the factors for the algebraic ſummation in the latter.

And now, Gentle Reader, for your farther acquaintance with *men, manners, and things*, I ſhall preſent you (by way of concluſion) with the tranſlation of a few pages from the *Acta Eruditorum* of Leipzig, for September, 1762 (p. 458.); and when you have well digeſted, and ſtrictly compared it with that Gentleman's ſecond *Memoir* (p. 23, &c.), particularly the ninth and tenth articles thereof, I would adviſe you to annihilate thoſe wonderful *inſignia* with which that memoir is ſo profuſely embelliſhed; not only as a juſt ſacrifice to the *manes* of John Francis de Tuſchis de Fagnano and his venerable father, but that the ſcurrilous and ill-natured may not thereby take occaſion to ſay, that a *Britiſh Mathematician* had aſcribed the honour of a diſcovery to himſelf, which he had "*baſely pilfered*" from a *Foreigner*.

A Demonſtration of a Theorem propoſed in the Leipſig Acts, for January 1754 (p. 40.), by *Archdeacon*

JOHN FRANCIS DE TUSCHIS DE FAGNANO,

Ex Sancti Honorii Marchionibus.

(Tranſlated from the *Acta Eruditorum*, 1762. p. 458.)

'IN the above acts Geometricians are invited to demonſtrate the following theorem,——If in the ellipſis AEBF (Fig. 4.) whoſe principal axes are AB and EF, any diameter be drawn as XY, and its ſemiconjugate CZ produced till CV is $=$ CA, and from V a perpendicular be drawn to AC, the ellipſis will be ſo cut in the point S that the difference of the arcs XAS and YFS may be aſſigned geometrically. For if from X the perpendicular XQ be drawn to CZ, we ſhall have YFS$-$XAS $=$ 2 CQ.

'In order to facilitate the demonſtration of this theorem I ſhall premiſe the following *lemma*;

In the above ellipſis let CA $= a$, EC $= b$, CM $= x$; then, if the abciſs CP (z) be ſuppoſed to be ſo taken that its value may be expreſſed by $\frac{a\sqrt{a^2-x^2}}{\sqrt{a^2-\frac{a^2-b^2}{a^2}x^2}}$, I ſay

the

the arc EX — arc AT will be expreſſed by $\frac{\sqrt{a^2-b^2}\cdot xz}{a^2} = \frac{\sqrt{a^2-b^2}\times x\sqrt{a^2-x^2}}{a^2\sqrt{a^2-\frac{a^2-x^2}{a^2}x^2}}$.

'My Father was the firſt who diſcovered this property, which, with its demonſtration, may be ſeen in the Italian Diary *(Diarium Eruditorum Italiæ)* Vol. 26. for the year 1716, printed at Venice; being contained in the ſixth article thereof, intituled, *A Theorem, from whence may be deduced a new meaſure of the arcs of the Ellipſe, Hyperbola, and Cycloid* †. It is alſo found in the ſecond volume of his own works, p. 336. printed at Peſaro (in Italy) 1750.

'Demonſtration of the Theorem.

'Having drawn VT, and (through M) XH, ⊥ to AB, as alſo XQ ⊥ to CZ, we have *per* conics (retaining the above notation) $CO = \sqrt{a^2-x^2}$, $CZ = \sqrt{a^2-\frac{a^2-b^2}{a^2}x^2}$, and $CL = \frac{a^2-b^2}{a}x$; and becauſe of the parallels VP, ZO, ZC: $CO :: VC : PC = \frac{a\sqrt{a^2-x^2}}{\sqrt{a^2-\frac{a^2-b^2}{a^2}x^2}}$; alſo from the ſimilar triangles CPV, CQL,

$CV\,(CA) : CP :: CL : CQ = \frac{a^2-b^2}{a^2} \times \frac{x\sqrt{a^2-x^2}}{\sqrt{a^2-\frac{a^2-b^2}{a^2}x^2}}$ * = (per lemma)

arc EX — arc AT ‡. Now the arcs XE, FY, HF are evidently equal to each other, and therefore 2 arc EX = arc HY; and the arc TAS is = 2 arc AT; conſeq. arc YH — arc TAS is = 2 CQ; to which adding arc HS = arc XT, and tranſpoſing, becomes arc YH + arc HS — arc TAS — arc TX = arc YFS — arc XSA = 2 CQ. Q. E. D.

'Coroll. 1. To the point X draw the tangent XN, and CK ⊥ thereto, ſo will XN be parallel to the ſemidiameter CV, and KX evidently = CQ; conſeq. the arc YFS — arc XAS is = 2 KX.

'Cor. 2. When z is $= \frac{a\sqrt{a^2-x^2}}{\sqrt{a^2-\frac{a^2-b^2}{a^2}x^2}}$, we have $a^2z^2 - \frac{a^2-b^2}{a^2}x^2z^2 = a^4 - a^2x^2$,

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† This theorem, with its application to the rectifying of the Elliptic, Hyperbolic, and Cycloidal arcs (a copy of which from the above Diary I have been lately favoured with) will be inſerted in the ſecond edition of my *Rationale of Circulating Numbers*, which will ſoon be publiſhed, with *additions*.

* See Memoirs, art. 14.

‡ Mr. Landen has alſo diſcovered that the arc EX — arc AT is = (CQ) the tang. XK, ſee Mem. art. 9. l

conſeq.

conſeq. $a^2 \times \overline{z^2+x^2-a^2}$ is $= \frac{a^2-b^2}{a^2} x^2 z^2$; and from hence ariſes $\frac{a^2\sqrt{z^2+x^2-a^2}}{\sqrt{a^2-b^2}} =$

$xz = \frac{ax\sqrt{a^2-x^2}}{\sqrt{a^2-\frac{a^2-b^2}{a^2}x^2}}$, as alſo from the above demon. arc YFS − arc XAS $=$

$\frac{2\cdot\sqrt{a^2-b^2}}{a} \times \sqrt{z^2+x^2-a^2}$. Let now F be the focus of the ellipſe (Fig. 5.) then will $CF = \sqrt{a^2-b^2}$, and by ſubſtituting the *linear* values in this laſt equation we have

arc YFS − arc XAS $= \frac{2CF}{CA} \times \sqrt{CM^2+CP^2-CA^2} = \frac{2CF}{CA} \times \sqrt{CM^2-PV^2} =$

$\frac{2CF}{CA} \times \sqrt{AOB-APB}$.

‘ Cor. 3. If in the analytical value of CQ we ſubſtitute the correſponding linear values, there ariſeth $CQ = \frac{CF^2}{CA^2} \times \frac{CM\cdot CO}{CZ}$. Now from the *foci* F, *f*, draw FX, *f*X, and produce the ſemiaxis CE till it meet the tangent NX in R; and let XI be the radius of curvature of the ellipſe at the point X. Then (per conics) we have theſe four equations

$$CZ = \sqrt{FX\cdot fX} = \sqrt{RX\cdot XN} = \sqrt{IX\cdot XQ} = \sqrt[3]{CA\cdot CE\cdot IX}.$$

Therefore alſo the following

$$\frac{CF^2}{CA^2} \times \frac{MCO}{CZ} = \begin{cases} \frac{CF^2}{CA^2} \times \frac{MCO}{\sqrt{FX\cdot fX}}, \\ \frac{CF^2}{CA^2} \times \frac{MCO}{\sqrt{RX\cdot XN}}, \\ \frac{CF^2}{CA^2} \times \frac{MCO}{\sqrt{IX\cdot XQ}}, \\ \frac{CF^2}{CA^2} \times \frac{MCO}{\sqrt[3]{CA\cdot CE\cdot IX}}, \end{cases}$$

which being multiplied by 2, are each equal to the *arc* YFS − *arc* XAS. And if in this laſt equation there be ſubſtituted $\frac{CA^2\cdot LX^3}{CE^4}$ for the radius of curvature IX, there ariſes *arc* YFS − *arc* XAS $= \frac{\overline{2CF}^2}{CA^3} \times \frac{CE\cdot CM\cdot CO}{LX}$.

‘ Cor. 4. If the difference of the elliptic arcs be required to be a maximum, we have, by making the fluxion of (2CQ) $\frac{2\times\sqrt{a^2-b^2}}{a^2} \times \frac{x\sqrt{a^2-x^2}}{\sqrt{a^2-\frac{a^2-b^2}{a^2}x^2}} = 0$, and

reducing

reducing, $x = \frac{a^{\frac{3}{2}}}{\sqrt{a+b}}$; which value of x being ſubſtituted in the equation $z = a\sqrt{a^2-x^2}$, we find $z = x$. If therefore we make $x = \frac{a^{\frac{3}{2}}}{\sqrt{a+b}}$, and take $z = x$, the difference of the arcs YFS, XAS, will be a maximum.

'Cor. 5. (Fig. 6.) When the point T falls in X, and theref. S in H, the equation becomes, arc YH − arc XAH $= \frac{2 \cdot \overline{a^2-b^2}}{a^3} \times \frac{a^3}{a+b} = 2 \cdot \overline{a-b}$ *.

'Cor. 6. (Fig. 4.) In the circumſtances mentioned in the two laſt corollaries, the tangent XK will be a maximum, *per Cor.* 1.

'Cor. 7. From Cor. 5. (or rather from the lemma) we have, arc EX − arc AX $= a-b$; to which adding the quadrantal arc AE, there ariſes, arc AE + arc EX − arc AX $= a-b$ + arc AE, that is, 2 arc EX $= a-b$ + AE †.

'Cor. 8. (Fig. 6.) Having drawn Xx, HY, parallel to AB, we have from the preceding corollary, 2 arc HF $= a-b$ + AF; and therefore $2 \times$ arc EX + arc HY $= 2 \cdot \overline{a-b}$ + ſemiellipſe EAF, that is, arc X$E x$ + arc HFY = AB − EF + EAF. Which equation, multiplied by 2, exhibits the following theorem,

The whole periphery of the ellipſis EAFBE *is equal to*

$$2 \times \text{arc XE}x + \text{arc HFY} + \text{EF} - \text{AB}.$$

'Cor. 9. (Fig. 7.) In the caſe when the conjugate ſemidiameters CC, Cg, are equal, we have CM $(x) = \frac{a}{\sqrt{2}}$, and therefore CP $= z = \frac{a\sqrt{a^2-x^2}}{\sqrt{a^2 - \frac{a^2-b^2}{a^2}x^2}} = \frac{a^2}{\sqrt{a^2+b^2}}$; hence *(per lemma)* arc EG − arc AT $= \frac{a^2-b^2}{\sqrt{2 \cdot a^2+b^2}} = \frac{a^2-b^2}{2CG}$.

'Cor. 10. From the equations in the Coroll. 5. and 9. ariſe the following analogies, arc EX − arc AX : arc EG − arc AT :: $a-b : \frac{a^2-b^2}{2CG}$ :: 2CG : $a-b$ :: CG + Cg : AC + CE.

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* And therefore arc FH − arc AH $= a-b =$ KX (cor. 6.) ſee Memoirs, art. 5.

† This very ſame property alſo has Mr. Landen diſcovered, ſee Mem. art. 10.!!

FINIS.

ERRATA.

In Σ N° 271. l. 2. for gp, r. gq. In Σ N° 275. for rad. r, r. rad. r. In F N° 288. for $n-r$, r. $n+r$.
In Σ N° 293. l. 3. for 2 C, r. cC. In Σ N° 295. write n over the exterior radical ſigns.
In Σ N° 300. for $= b$, r. $= 2b$. In F N° 311. for $5bc^3$, r. $4bc^3$.
In F N° 317. for $\frac{x^2}{3}$, r. $\frac{x^3}{3}$. In F N° 320. for $\frac{x^5}{2^2}$, r. $\frac{x^5}{5^2}$

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